The Higher Spin Currents in the $\mathcal{N}=1$ Stringy Coset Minimal Model

Changhyun Ahn

Department of Physics, Kyungpook National University, Taegu 702-701, Korea

ahn@knu.ac.kr

Abstract

We reconsider the $\mathcal{N}=1$ supersymmetric extension of W_3 algebra which was studied previously. This consists of seven higher spin supercurrents (fourteen higher spin currents in components) as well as the $\mathcal{N}=1$ stress energy tensor of spins $(\frac{3}{2},2)$. So far, the complete expressions for the higher spin currents are not known.

In this paper, we construct them explicitly both in the $c=4$ eight free fermion model and in the supersymmetric coset model based on $(A_2^{(1)} \oplus A_2^{(1)}, A_2^{(1)})$ at level $(3,k)$. By acting the above spin-$\frac{3}{2}$ current on the higher spin-3 Casimir current, one generates its fermionic partner, the higher spin-$\frac{5}{2}$ current and they can be combined as a first higher spin supercurrent with spins $(\frac{5}{2},3)$. Then by computing the operator product expansions(OPE) between the higher spin supercurrent and itself, one can generate the next two higher spin supercurrents with spins $(\frac{7}{2},4)$ and $(4,\frac{9}{2})$. Moreover, the other two higher spin supercurrents with spins $(4,\frac{9}{2})$ and $(\frac{9}{2},5)$ are generated by calculating the OPE between the first higher spin supercurrent with spins $(\frac{5}{2},3)$ and the second higher spin supercurrent with spins $(\frac{7}{2},4)$. Finally, the remaining two higher spin supercurrents with spins $(\frac{11}{2},6)$ and $(6,\frac{13}{2})$, from the first higher spin supercurrent and the fourth higher spin supercurrent with spins $(4,\frac{9}{2})$, are obtained similarly.

1 Introduction

The duality between the W_N minimal model in two dimensional conformal field theories and the higher spin theory of Vasiliev on the AdS_3 has been proposed by Gaberdiel and Gopakumar [1, 2]. The W_N minimal model conformal field theory is dual, in the 't Hooft $\frac{1}{N}$ expansion, to the higher spin theory coupled to one complex scalar. One of the level for spin-1 WZW current in the conformal field theory is fixed by 1 and the other is given by the positive integer k.

By generalizing the level 1 for the spin-1 current to the arbitrary positive integer l, one has more general coset theories. The spin-3 current in these general cosets was constructed in [3] and the spin-4 current was found recently in [4]. The central charge grows like N^2 by defining the 't Hooft limit with $k, l, N \to \infty$ with the appropriate relative ratios held finite [5] [1]. The particular case where $k = l = N$ was studied in the context of two dimensional gauge theory coupled to adjoint fermions in [6].

It is known, in [7], that the cosets can be supersymmetric if one of the levels is equal to N(i.e., $l = N$ with arbitrary k). Furthermore, the first model with $k = 1$ in this series (with arbitrary l and N) has bosonic W_N symmetry. The supersymmetric version of W_N algebra with $k = 1, l = N$ was studied in [8]. On the other hand, by taking $N = 3 = l$ with arbitrary k, one expects to have a supersymmetric extended W_3 symmetry.

In this paper, we reconsider the following coset minimal model studied in [9] previously, with arbitrary k and $l = N = 3$,

$$\frac{\widehat{SU}(3)_k \oplus \widehat{SU}(3)_3}{\widehat{SU}(3)_{k+3}}. \tag{1.1}$$

The central charge can be obtained and it is given by

$$c = 4\left[1 - \frac{18}{(k+3)(k+6)}\right], \qquad k = 1, 2, \cdots. \tag{1.2}$$

For $k = 1 (c = \frac{10}{7})$ with (1.2), the explicit higher spin current as well as the superconformal generators were constructed in [10] and they are given in terms of the WZW currents of above coset (1.1). This extended algebra coincides with the "minimal" super W_3 algebra found in [11] where there exist two extra higher spin currents of spin-$\frac{5}{2}$ and spin-3. In [9], some of the higher spin currents in $c = 4$, which is equivalent to $k \to \infty$ limit in (1.2), free fermion model and the generic $c < 4$ supersymmetric coset models based on (1.1) were found and the existence for eight supercurrents via character technique was described. In [12], the possible

[1] These general cosets are called by "stringy cosets" in [5].

application in the context of minimal model holography [1, 2] was described. If one considers the discrete series for the $\mathcal{N}=2$ superconformal minimal models with $c=\frac{3m}{(m+2)}$, one observes that $c=\frac{8}{3}$ (i.e., $k=3$) corresponds to $m=16$.

The main thing in the extended algebra on the basis of the Casimir construction [13, 3] for level 1 WZW models for simply laced Lie algebras is to identify the complete set of independent generating currents. We would like to find out the complete set of currents proposed in [9] explicitly and some of the algebra they satisfy. The starting point is to begin with the bosonic spin-3 current found in [13, 3] with fixed $N=3$ and finite or infinite k. Of course, it is known that the $\mathcal{N}=1$ stress energy tensor consisting of spin-2 and spin-$\frac{3}{2}$ currents can be obtained from the usual Sugawara construction. All the higher spin currents, then, can be obtained from the spin-$\frac{3}{2}$ current and spin-3 current. These two currents generate the higher spin currents [2].

That is, by acting the above spin-$\frac{3}{2}$ current on the higher spin-3 current (by computing the operator product expansion (OPE) between them), one generates its fermionic partner, the higher spin-$\frac{5}{2}$ current in the right hand side of OPE and they can be combined as a first higher spin $\mathcal{N}=1$ supercurrent with spins $(\frac{5}{2},3)$ [3]. By construction, the OPE between the composite fields can be computed through the basic fundamental OPEs between the bosonic and fermionic WZW currents. Then by computing the OPE between these two higher spin currents of spin-$\frac{5}{2}$ and spin-3 (therefore one should compute three nontrivial OPEs in the component approach), one can generate the next two higher spin $\mathcal{N}=1$ supercurrents with spins $(\frac{7}{2},4)$ and $(4,\frac{9}{2})$ that appear in the singular terms of the OPE. The explicit forms for these in terms of WZW currents for $c=4(k\to\infty)$ model were already present in [9]. For the arbitrary central charge(i.e., finite k), the explicit forms are not known so far. Moreover, the other two higher spin $\mathcal{N}=1$ supercurrents with spins $(4,\frac{9}{2})$ and $(\frac{9}{2},5)$ are generated by calculating the OPE between the first higher spin supercurrent with spins $(\frac{5}{2},3)$ and the second higher spin supercurrent with spins $(\frac{7}{2},4)$ by repeating similar procedures. There exist also various quasi-primary field currents that can be written in terms of known higher spin currents and the $\mathcal{N}=1$ superconformal currents. Finally, the remaining two higher spin $\mathcal{N}=1$ supercurrents with spins $(\frac{11}{2},6)$ and $(6,\frac{13}{2})$, from the first higher spin supercurrent and the fourth higher spin supercurrent with spins $(4,\frac{9}{2})$, are obtained similarly. This will

[2]These higher spin currents are primary under the stress energy tensor and they do transform similarly under the spin-$\frac{3}{2}$ current. Combining these two, the higher spin currents are superprimary under the $\mathcal{N}=1$ stress energy tensor.

[3]Although we will explain the notation in section 2, let us describe the convention we use here. The fermionic current of spin $\frac{5}{2}$ and its superpartner bosonic current of spin 3 can be written in terms of a single $\mathcal{N}=1$ supercurrent and we simply denote this by specifying the spins in this way. Similarly, one denotes the $\mathcal{N}=1$ superconformal supercurrent by $(\frac{3}{2},2)$.

produce the most complicated computations.

How does one compute the OPE explicitly and extract the correct primary currents in the right hand side of OPE? In general, the OPE [14, 15, 16] between two quasi-primary fields $\Phi_i(z)$ with spin-h_i and $\Phi_i(w)$ with spin-h_j(the spins are positive integer or half-integer) has the following form [4]

$$\Phi_i(z)\,\Phi_j(w) = \frac{1}{(z-w)^{h_i+h_j}}\,\gamma_{ij} + \sum_k C_{ijk} \sum_{n=0}^{\infty} \frac{1}{(z-w)^{h_i+h_j-h_k-n}} \left[\frac{1}{n!} \frac{\Gamma(h_i-h_j+h_k+n)}{\Gamma(h_i-h_j+h_k)} \frac{\Gamma(2h_k)}{\Gamma(2h_k+n)} \right] \partial^n \Phi_k(w). \quad (1.3)$$

The γ_{ij} corresponds to a metric on the space of quasi-primary fields. The quantity C_{ijk} appears in the three-point function between the quasi-primary fields $\Phi_i(z)$, $\Phi_j(z)$ and $\Phi_k(z)$. The index k specifies all the quasi-primary fields occurring in the right hand side. The descendant fields for quasi-primary field $\Phi_k(w)$ are (multiple) derivatives of $\Phi_k(w)$:$\partial^n \Phi_k(w)$. Furthermore, the coefficient functions are written in terms of various Gamma function and they depend on the spins and the number of derivatives [5]. As long as $n \geq h_i+h_j-h_k$, one sees the regular terms. Otherwise ($n \leq h_i+h_j-h_k$), the singular terms exist. By noting that the Gamma function has the following property $\frac{\Gamma(h_i-h_j+h_k+n)}{\Gamma(h_i-h_j+h_k)} = (h_i-h_j+h_k)(h_i-h_j+h_k+1)\cdots(h_i-h_j+h_k+n-1)$ which is a Pochhammer function, for $h_i-h_j+h_k \leq 0$ (i.e., the spin of $\Phi_j(z)$ is greater than the sum of the spin of $\Phi_i(z)$ and the spin of $\Phi_k(z)$), the summation over n terminates to a finite summation. For example, when $(h_i-h_j+h_k+n-1)$ is vanishing for particular $n=n_0$, then the coefficient for $n > n_0$ always contains this vanishing value $(h_i-h_j+h_k+n_0-1)$. For the factor $\frac{\Gamma(2h_k)}{\Gamma(2h_k+n)}$, there is no zero for the positive h_k and n. Note that for different k's (different quasi-primary fields), the corresponding spin h_k's can be equal to each other. That is, for given singular terms, one has several different quasi-primary fields of same spin. we will see this feature in next sections.

It is nontrivial to find out the possible quasi-primary or primary fields $\Phi_k(w)$ in the right hand side. For lower higher spin quasi-primary or primary fields, the number of quasi-primary or primary fields is very limited but as the spin increases, the number of quasi-primary or primary fields also increases. As mentioned before, in general, the quasi-primary fields in (1.3)

[4]The OPE of the stress energy tensor with the quasi-primary field does not have third-order pole. The primary field is also a quasi-primary field because it satisfies with this condition for the quasi-primary field. In general, the OPE between the stress energy tensor and quasi-primary field can have any nonzero singular term(s) with order $n > 3$. Of course, for the primary field, the trivial vanishing higher order singular terms with order $n \geq 3$ occur. In this paper, we restrict to the definition of quasi-primary field as follows: what we mean by quasi-primary field is the one which does not contain the primary field. When we say quasi-primary field, the nonvanishing higher order (greater than 3) singular terms exist.

[5]It seems to me that the expression (1.3) initiated by W. Nahm appeared in [17, 18] but I do not have this copy at the moment. I think one can find more details there.

are given in terms of composite fields between WZW currents of integer or half-integer spins. By construction, one obtains the OPEs from the basic fundamental OPEs between WZW currents. Then one gets all the singular terms for given spins h_i and h_j. It is better to analyze the most singular term and then do the next lower singular terms because once the lowest quasi-primary field (or primary field) is found, then its descendant structure is completely fixed according to (1.3). Then after the nontrivial highest singular term is analyzed (the quasi-primary field or primary field appears in the right hand side), then the next singular term contains the descendant field for the previous quasi-primary or primary fields and the remaining terms contain the new quasi-primary fields or primary fields according to (1.3).

In general, it is nontrivial to find these new fields. However, they should transform as quasi-primary or primary fields under the spin-2 current as we explained before. Therefore, one should compute the OPEs between the spin-2 and "the remaining terms" in the given singular terms. In general, this will include all the higher singular terms where the order is greater than 3. We have to figure out what kind of quasi-primary or primary fields occur. By taking the possible quasi-primary fields with undetermined coefficient functions, one should write down the remaining terms in terms of these quasi-primary or primary fields. Then the unknown coefficient functions can be fixed by using the condition that the third-order pole should vanish. Furthermore, one should compute the OPEs between the spin-$\frac{3}{2}$ current and "the remaining terms" in order to see the complete structure of the possible quasi-primary fields. For the quasi-primary fields, there is no constraint on the singular terms but for the primary fields, the OPE between the spin-$\frac{3}{2}$ current and the primary current should contain either the first-order pole or the second- and first-order poles. This implies that at least the OPE of spin-$\frac{3}{2}$ and the primary field should not contain the higher order singular terms where the order is greater than 2. We will see the details in next sections.

In section 2, after the $\mathcal{N}=1$ superconformal algebra and the "minimal" $\mathcal{N}=1$ super W_3 algebra [11] are reviewed, the higher spin currents in $c=4$ free fermion model are described.

In section 3, the higher spin currents in $c<4$ coset model are constructed explicitly.

In section 4, we summarize what we have found in this paper and comment on the future directions.

In the Appendices, we will present some OPEs relevant to the sections 2 and 3.

The $\mathcal{N}=2$ supersymmetric extension of [1, 2] with higher spin AdS_3 supergravity has been studied in [19, 20, 21, 22, 23, 24, 25] where the dual conformal field theory is given by $\mathcal{N}=2$ $\mathbf{CP}^N$ Kazama-Suzuki model in two dimensions. Recently, in [26], the $\mathcal{N}=1$ minimal model holography is described.

One can find that all the relevant works, along the line of [1, 2], are presented in the recent

review papers [5, 27] and also further works are given in [28]-[40].

The mathematica package [41] is used all the time.

2 The higher spin currents in the $c = 4$ eight free fermion model

2.1 The $\mathcal{N} = 1$ superconformal algebra: review

Let us describe the $\mathcal{N} = 1$ supersymmetric extension of Virasoro algebra. The $\mathcal{N} = 1$ superconformal algebra is generated by the $\mathcal{N} = 1$ super stress energy tensor of spin-$\frac{3}{2}$ [42],

$$\hat{T}(Z) = \frac{1}{2}G(z) + \theta\, T(z), \tag{2.1}$$

where $Z = (z, \theta)$ is a complex supercoordinate, $T(z)$ is the usual bosonic stress energy tensor of spin-2 and $G(z)$ is its fermionic superpartner of spin-$\frac{3}{2}$. The superconformal algebra, in components, is summarized by the three OPEs as follows. The OPE between the fermionic field of spin-$\frac{3}{2}$ and itself reads as

$$G(z)\, G(w) = \frac{1}{(z-w)^3}\,\frac{2}{3}c + \frac{1}{(z-w)}\, 2T(w) + \cdots, \tag{2.2}$$

where the right hand side of this OPE contains the bosonic stress energy tensor as well as the central term. It is obvious that there is no second-order singular term because there is no spin-1 field. The standard OPE between the bosonic stress energy tensor and itself (i.e., Virasoro algebra) is given by

$$T(z)\, T(w) = \frac{1}{(z-w)^4}\,\frac{c}{2} + \frac{1}{(z-w)^2}\, 2T(w) + \frac{1}{(z-w)}\, \partial T(w) + \cdots, \tag{2.3}$$

and finally, the fermionic field is primary with respect to $T(w)$, i.e.,

$$T(z)\, G(w) = \frac{1}{(z-w)^2}\,\frac{3}{2}G(w) + \frac{1}{(z-w)}\, \partial G(w) + \cdots. \tag{2.4}$$

Of course, the OPE $G(z)\, T(w)$ can be obtained from (2.4) in standard way. Then, the $\mathcal{N} = 1$ superconformal algebra is represented by (2.2), (2.3) and (2.4) or its $\mathcal{N} = 1$ single OPE with (2.1). One can also express these OPEs using the (anti)commutator relations for the modes of $T(z)$ and $G(z)$, as usual. For the integer mode of $G(z)$, one has Ramond algebra and for the half-integer mode of $G(z)$, one has Neveu-Schwarz algebra. According to the definition of quasi-primary field, the stress energy tensor $T(z)$ is a quasi-primary field. The coefficients 2 in (2.2), 2 in (2.3) and $\frac{3}{2}$ in (2.4) play the role of C_{ijk} in (1.3) and the central terms in (2.2)

and (2.3) correspond to γ_{ij} in (1.3). Moreover, the relative coefficients $\frac{1}{2}$ in (2.3) and $\frac{2}{3}$ in (2.4) appearing in the first-order singular term coincide with the general expression given in (1.3). In the Appendix A, we present more details on the coefficient functions.

2.2 The "minimal" $\mathcal{N}=1$ super W_3 algebra where $c=\frac{10}{7}$: review

The simplest extension of the previous $\mathcal{N}=1$ superconformal algebra is to add a single higher spin superprimary current of spin-$\frac{5}{2}$

$$\hat{W}(Z)=\frac{1}{\sqrt{6}}U(z)+\theta\,W(z), \tag{2.5}$$

where $W(z)$ is a bosonic spin-3 current and $U(z)$ is a fermionic spin-$\frac{5}{2}$ current. They are primary fields with respect to the stress energy tensor $T(z)$, like as (2.4) and furthermore, the spin-$\frac{3}{2}$ current $G(z)$ transforms $U(w)$ into $W(w)$ and vice versa (fermion goes to boson and boson goes to fermion).

$$G(z)\,U(w)=\frac{1}{(z-w)}\,\sqrt{6}\,W(w)+\cdots, \tag{2.6}$$

and

$$G(z)\,W(w)=\frac{1}{(z-w)^2}\frac{5}{\sqrt{6}}\,U(w)+\frac{1}{(z-w)}\frac{1}{\sqrt{6}}\,\partial U(w)+\cdots. \tag{2.7}$$

This implies that once any component field of (2.5) is found, then its superpartner can be determined automatically from (2.6) or (2.7). Again, the relative coefficient $\frac{1}{5}$ appearing the first-order singular term in (2.7) comes from the general expression in (1.3). The role of the spin-$\frac{3}{2}$ current $G(z)$ is very important and we will use this property all the time.

By assuming that the OPE between the additional supercurrent (2.5) and itself does not generate any new superprimary current (i.e., the "minimal" extension), one can write down the possible structures in the right hand side of the OPE. The unknown coefficient functions can be determined by the so-called Jacobi identities for normal ordered graded commutators of the supercurrents $\hat{T}(Z)$ and $\hat{W}(Z)$. It turns out that the "minimal" $\mathcal{N}=1$ super W_3 algebra is associative for $c=\frac{10}{7}$ where the unitary representation exists and $c=-\frac{5}{2}$ with nonunitary representation [11]. For $c=\frac{10}{7}$, the three OPEs in components are summarized as follows. The OPE between the bosonic spin-3 current and itself leads to

$$\begin{aligned} W(z)\,W(w) &= \frac{1}{(z-w)^6}\,\frac{10}{21}+\frac{1}{(z-w)^4}\,2T(w)+\frac{1}{(z-w)^3}\,\partial T(w) \\ &+ \frac{1}{(z-w)^2}\left[\frac{3}{10}\partial^2 T+\frac{56}{51}\left(T^2-\frac{3}{10}\partial^2 T\right)\right](w) \\ &+ \frac{1}{(z-w)}\left[\frac{1}{15}\partial^3 T+(\frac{1}{2})\frac{56}{51}\partial\left(T^2-\frac{3}{10}\partial^2 T\right)\right](w)+\cdots, \end{aligned} \tag{2.8}$$

which is exactly the same as the Zamolodchikov's W_3 algebra for $c = \frac{10}{7}$, as we expected. The coefficient $\frac{1}{2}$ for the descendant field with spin-5 associated with the quasi-primary field of spin 4 in (2.8) comes from the general expression in (1.3). The coefficients $\frac{1}{2}, \frac{3}{20}$ and $\frac{1}{30}$ appearing in the descendant fields of stress energy tensor can be obtained similarly [6].

The OPE between the spin-3 and spin-$\frac{5}{2}$ is summarized by

$$\begin{aligned} W(z)\,U(w) &= \frac{1}{(z-w)^4}\,\frac{3}{\sqrt{6}}G(w) + \frac{1}{(z-w)^3}\,(\frac{2}{3})\frac{3}{\sqrt{6}}\partial G(w) \\ &+ \frac{1}{(z-w)^2}\left[(\frac{1}{4})\frac{3}{\sqrt{6}}\partial^2 G + \frac{77\sqrt{6}}{187}\left(GT - \frac{1}{8}\partial^2 G\right)\right](w) \\ &+ \frac{1}{(z-w)}\left[(\frac{1}{15})\frac{3}{\sqrt{6}}\partial^3 G + (\frac{4}{7})\frac{77\sqrt{6}}{187}\partial\left(GT - \frac{1}{8}\partial^2 G\right)\right. \\ &+ \left.\frac{4\sqrt{6}}{17}\left(\frac{4}{3}T\partial G - G\partial T - \frac{4}{15}\partial^3 G\right)\right](w) + \cdots. \end{aligned} \tag{2.9}$$

We intentionally write down the coefficients $\frac{2}{3}, \frac{1}{4}$ and $\frac{1}{15}$ for the descendant fields for the spin-$\frac{3}{2}$ field in the right hand side (2.9) in order to emphasize that they can be determined from (1.3). There are two quasi-primary fields of spin-$\frac{7}{2}$ and spin-$\frac{9}{2}$. The coefficient $\frac{4}{7}$ appearing in the descendant field for the former can be obtained from the general formula. The first-order singular term consists of the descendant field $\partial^3 G(w)$ for $G(w)$, the descendant field $\partial(GT - \frac{1}{8}\partial^2 G)(w)$ for spin-$\frac{7}{2}$ quasi-primary field $(GT - \frac{1}{8}\partial^2 G)(w)$ and the quasi-primary field $(\frac{4}{3}T\partial G - G\partial T - \frac{4}{15}\partial^3 G)(w)$ of spin-$\frac{9}{2}$. In other words, three independent terms, characterized by $\partial^3 G(w), \partial GT(w)$ and $G\partial T(w)$ are rewritten in terms of above three terms [7].

Finally, the spin-$\frac{5}{2}$ and spin-$\frac{5}{2}$ OPE is given by

$$\begin{aligned} U(z)\,U(w) &= \frac{1}{(z-w)^5}\,\frac{4}{7} + \frac{1}{(z-w)^3}\,2T(w) + \frac{1}{(z-w)^2}\,\partial T(w) \\ &+ \frac{1}{(z-w)}\left[\frac{3}{10}\partial^2 T + \frac{63}{68}\left(T^2 - \frac{3}{10}\partial^2 T\right) + P_4^{uu}\right](w) + \cdots. \end{aligned} \tag{2.10}$$

The relative coefficients $\frac{1}{2}$ and $\frac{3}{20}$ appearing in the descendant fields for the stress energy tensor $T(w)$ can be analyzed previously and provide the correct values. In the first-order

[6] The quasi-primary field has the following OPE with the stress energy tensor $T(z)$ $(TT - \frac{3}{10}\partial^2 T)(w) = \frac{1}{(z-w)^4}\,(\frac{22}{5} + c)\,T(w) + \mathcal{O}((z-w)^{-2})$ where there is no third-order singular term.

[7] One can easily get the following OPEs in order to check whether they are really quasi-primary fields or not. They are $T(z)$ $(GT - \frac{1}{8}\partial^2 G)(w) = \frac{1}{(z-w)^4}\,\frac{37}{8}\,G(w) + \mathcal{O}((z-w)^{-2})$ and $T(z)$ $(\frac{4}{3}T\partial G - G\partial T - \frac{4}{15}\partial^3 G)(w) = -\frac{1}{(z-w)^5}\,\frac{33}{5}G(w) - \frac{1}{(z-w)^4}\,(-\frac{1}{3})\,\frac{33}{5}\partial G(w) + \mathcal{O}((z-w)^{-2})$. The coefficient $-\frac{1}{3}$ coincides with the general expression (1.3) by substituting $h_i = 2, h_j = \frac{9}{2}, h_k = \frac{3}{2}$, and $n = 1$. The minus sign is due to the fact that $h_i - h_j + h_k = -1$. In the Appendices B, C we present other quasi-primary fields and their OPEs between $T(z)$ or $G(z)$.

singular term, there exist a quasi-primary field $(T^2 - \frac{3}{10}\partial^2 T)(w)$ and primary field of spin-4 [10] given by

$$P_4^{uu}(w) = \frac{21}{17}\left[-\frac{7}{10}\partial^2 T + \frac{7}{12}\left(T^2 - \frac{3}{10}\partial^2 T\right) + G\partial G\right](w), \tag{2.11}$$

where the central charge $c = \frac{10}{7}$ is used. For the general c, the coefficient $\frac{7}{12}$ should be replaced by $\frac{17}{(22+5c)}$. Note that this primary field is not an additional field because this can be obtained from the currents $T(w)$ and $G(w)$. In this example, at the first-order singular term of (2.10), there are two kinds of quasi-primary fields. In the language of (1.3), for fixed h_k, there are two degeneracies. More precisely, one is spin-4 quasi-primary field and the other is spin-4 primary field (2.11). We will see that this primary field (2.11) has unusual OPE with the above spin-$\frac{3}{2}$ current $G(z)$ in the next subsection.

2.3 The $c = 4$ free fermion model

Then how does one obtain the consistent generalization of the above minimal extension of W_3 algebra for arbitrary central charge? We will consider the particular supersymmetric coset model introduced in the introduction. Before we will go into the details, we describe its particular limit where one sees all the algebraic structures.

Let us consider the eight fermion fields ψ^a of spin-$\frac{1}{2}$ where the $SU(3)$ adjoint index a runs from 1 to $8(=3^2-1)$. In this paper, N is fixed by 3. The OPE of this fermion field is given by

$$\psi^a(z)\ \psi^b(w) = -\frac{1}{(z-w)}\,\frac{1}{2}\delta^{ab} + \cdots. \tag{2.12}$$

Let us define spin-1 Kac-Moody current $J^a(z)$ as

$$J^a(z) = f^{abc}\psi^b\psi^c(z). \tag{2.13}$$

Then it is easy to compute the OPE between this spin-1 current and itself and it leads to

$$J^a(z)\ J^b(w) = -\frac{1}{(z-w)^2}\frac{3}{2}\delta^{ab} + \frac{1}{(z-w)}f^{abc}J^c(w) + \cdots, \tag{2.14}$$

where the level is given by 3. Similarly, one obtains

$$\psi^a(z)\ J^b(w) = \frac{1}{(z-w)}\,f^{abc}\psi^c(w) + \cdots. \tag{2.15}$$

In the context of (1.3), the above OPEs (2.12), (2.14) and (2.15) can be described easily.

For a given $\mathcal{N}=1$ super Kac-Moody algebra where $\hat{Q}^a(Z)=\sqrt{3}\,\psi^a(z)+\theta\ J^a(z)$, characterized by (2.12), (2.14) and (2.15), the $\mathcal{N}=1$ superconformal algebra is realized by the spin-2 current

$$T(z)=\psi^a\partial\psi^a(z) \tag{2.16}$$

and the spin-$\frac{3}{2}$ current

$$G(z)=-\frac{2}{3\sqrt{3}}\,\psi^a J^a(z). \tag{2.17}$$

They satisfy (2.2), (2.3) and (2.4) for $c=4$. Then the normalizations in (2.16) and (2.17) are fixed automatically.

Now it is ready to construct the higher spin currents. It is known, in [13], that the spin-3 current is described by the third order Casimir operator for $A_2^{(1)}=\widehat{SU}(3)$,

$$W(z)=\sqrt{\frac{2}{1215}}\,i\,d^{abc}J^aJ^bJ^c(z), \tag{2.18}$$

where d^{abc} is a completely symmetric traceless $SU(3)$ invariant tensor of rank 3 and spin-1 current is defined as (2.13). As we mentioned before, its fermionic superpartner of spin-$\frac{5}{2}$ can be determined from the relation (2.7) with (2.17) and (2.18)

$$U(z)=\sqrt{\frac{2}{375}}\,i\,d^{abc}\psi^aJ^bJ^c(z). \tag{2.19}$$

So far, the currents are given by the $\mathcal{N}=1$ superconformal generators and $\mathcal{N}=1$ W_3 supercurrent. However, one realizes that there appear the additional higher spin currents once one computes the OPEs between these currents.

In [9], the extra higher spin currents are given. Let us present them in $\mathcal{N}=1$ superspace with their components. We denote their spins in the subscript and we use the prime notation in order to describe the different field content with same spin

$$\begin{aligned}
\hat{W}(Z) &= \frac{1}{\sqrt{6}}U(z)+\theta\,W(z),\\
\hat{O}_{\frac{7}{2}}(Z) &= O_{\frac{7}{2}}(z)+\theta\,O_4(z),\\
\hat{O}_4(Z) &= O_{4'}(z)+\theta\,O_{\frac{9}{2}}(z),\\
\hat{O}_{4'}(Z) &= O_{4''}(z)+\theta\,O_{\frac{9}{2}'}(z),\\
\hat{O}_{\frac{9}{2}}(Z) &= O_{\frac{9}{2}''}(z)+\theta\,O_5(z),\\
\hat{O}_{\frac{11}{2}}(Z) &= O_{\frac{11}{2}}(z)+\theta\,O_6(z),\\
\hat{O}_6(Z) &= O_{6'}(z)+\theta\,O_{\frac{13}{2}}(z).
\end{aligned} \tag{2.20}$$

One can also introduce the normalization factor $\frac{1}{\sqrt{2h_{\hat{O}}+1}}$ in front of θ-independent term, like as $\hat{W}(Z)$. Some of the currents were found explicitly in [9]. We will complete the higher spin currents in terms of eight fermion fields $\psi^a(z)$. In particular, we will compute some OPEs between $\hat{W}(Z)$, $\hat{O}_{\frac{7}{2}}(Z)$ and $\hat{O}_{4'}(Z)$. There is no reason why we consider $\hat{O}_{4'}(Z)$ rather than $\hat{O}_4(Z)$. In doing these computations, one finds that the unknown higher spin currents arise in the right hand side of the OPE. We would like to construct the 12 higher spin currents in terms of $\psi^a(z)$ explicitly.

Let us consider the lower higher spin currents first.

- The construction of higher spin supercurrents $\hat{O}_{\frac{7}{2}}(Z)$ and $\hat{O}_4(Z)$

Let us first consider the OPE between the spin-$\frac{5}{2}$ currents. By using the defining equation (2.12), (2.15) and (2.19), one obtains all the singular terms. It turns out that the OPE reads

$$\begin{aligned} U(z)\, U(w) &= \frac{1}{(z-w)^5}\,\frac{8}{5} + \frac{1}{(z-w)^3}\, 2T(w) + \frac{1}{(z-w)^2}\,\partial T(w) \\ &+ \frac{1}{(z-w)}\left[\frac{3}{10}\partial^2 T + \frac{9}{14}\left(T^2 - \frac{3}{10}\partial^2 T\right) + P_4^{uu} + P_{4'}^{uu}\right](w) + \cdots, \end{aligned} \tag{2.21}$$

where the spin-4 primary field is given by

$$P_4^{uu}(z) = \frac{75}{407}\left[-\frac{7}{10}\partial^2 T + \frac{17}{42}\left(T^2 - \frac{3}{10}\partial^2 T\right) + G\partial G\right](z) \tag{2.22}$$

which is identically the same as the one in (2.11) with $c = 4$. Note that the new spin-4 primary field, compared to the OPE (2.10), arises as [10]

$$P_{4'}^{uu}(z) = -\frac{18}{407}\psi^a \partial^3 \psi^a(z) + \text{other lower order derivative terms}, \tag{2.23}$$

where we present only eight terms among 156 terms. In the context of (1.3), at the first-order singular term, there are triplet degeneracies for given spin-4 (quasi)primary fields. One way to see this extra new primary field (2.23) is to subtract $\frac{3}{10}\partial^2 T(w)$-term, $(TT - \frac{3}{10}\partial^2 T)(w)$-term with arbitrary coefficient, and spin-4 term (2.22) with undetermined coefficient, from the first-order singular term. Then these two unknown coefficients ($\frac{9}{14}$ and $\frac{75}{407}$) are fixed by the (quasi)primary condition. This spin-4 field (2.23) will provide some component field of $\mathcal{N}=1$ superprimary field $\hat{O}_{\frac{7}{2}}(Z)$ or $\hat{O}_4(Z)$.

Let us move on the next OPE between the spin-3 current (2.18) and the spin-$\frac{5}{2}$ current (2.19)

$$\begin{aligned} W(z)\, U(w) &= \frac{1}{(z-w)^4}\,\frac{3}{\sqrt{6}}G(w) + \frac{1}{(z-w)^3}\,(\frac{2}{3})\frac{3}{\sqrt{6}}\partial G(w) \\ &+ \frac{1}{(z-w)^2}\left[(\frac{1}{4})\frac{3}{\sqrt{6}}\partial^2 G + \frac{11\sqrt{6}}{37}\left(GT - \frac{1}{8}\partial^2 G\right) + O_{\frac{7}{2}}\right](w) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{(z-w)} \left[(\frac{1}{15}) \frac{3}{\sqrt{6}} \partial^3 G + (\frac{4}{7}) \frac{11\sqrt{6}}{37} \partial \left(GT - \frac{1}{8} \partial^2 G \right) + (\frac{4}{7}) \partial O_{\frac{7}{2}} \right. \\
&+ \left. \frac{4\sqrt{6}}{77} \left(\frac{4}{3} T \partial G - G \partial T - \frac{4}{15} \partial^3 G \right) + O_{\frac{9}{2}} \right] (w) + \cdots . \qquad (2.24)
\end{aligned}
$$

Compared to the minimal extension in previous subsection, there exist two new primary fields [10]. One is spin-$\frac{7}{2}$ field

$$
O_{\frac{7}{2}}(z) = -\frac{\sqrt{2}}{37} f^{abc} \psi^a \psi^b \partial^2 \psi^c(z) + \text{other first-order derivative terms} \qquad (2.25)
$$

where we present only the highest derivative terms among 99 terms. The other is spin-$\frac{9}{2}$ field

$$
O_{\frac{9}{2}}(z) = -\frac{4\sqrt{2}}{231} f^{abc} \psi^a \psi^b \partial^3 \psi^c(z) + \text{other lower order derivative terms} \qquad (2.26)
$$

which consists of 270 terms. The structure constant $\frac{11\sqrt{6}}{37}$ and $\frac{4\sqrt{6}}{77}$ in (2.24) are determined by (quasi)primary condition as before. In general, these are given in terms of the central charge and since we are considering $c = 4$ model, they are different from those in (2.9). The advantage of $c = 4$ model is that since we know the explicit form for the WZW currents, one can always compute the OPE and find out the singular terms. However, the construction in minimal extension is based on the assumption that there exist some extended generators whose realizations are not known. Therefore, one has to write down the possible structures with unknown coefficient functions in the right hand side of the OPE and then use Jacobi identities to fix them. The main things one should do in this subsection and next section is to arrange the known singular terms and extract all the possible (quasi)primary fields by using the relation (1.3) or the expression in the Appendix A, (A.1) and (A.2).

For the spin-3 current (2.18) and the spin-3 current OPE, one obtains

$$
\begin{aligned}
W(z)\, W(w) = &= \frac{1}{(z-w)^6} \frac{4}{3} + \frac{1}{(z-w)^4} 2T(w) + \frac{1}{(z-w)^3} \partial T(w) \\
&+ \frac{1}{(z-w)^2} \left[(\frac{3}{10}) \partial^2 T + \frac{16}{21} \left(T^2 - \frac{3}{10} \partial^2 T \right) + P_4^{ww} + P_{4'}^{ww} \right] (w) \\
&+ \frac{1}{(z-w)} \left[(\frac{1}{15}) \partial^3 T + (\frac{1}{2}) \frac{16}{21} \partial \left(T^2 - \frac{3}{10} \partial^2 T \right) + (\frac{1}{2}) \partial P_4^{ww} + (\frac{1}{2}) \partial P_{4'}^{ww} \right] (w) \\
&+ \cdots , \qquad (2.27)
\end{aligned}
$$

where the primary field of spin-4 is

$$
P_4^{ww}(z) = -\frac{48}{407} \left[-\frac{7}{10} \partial^2 T + \frac{17}{42} \left(T^2 - \frac{3}{10} \partial^2 T \right) + G \partial G \right] (z) \qquad (2.28)
$$

and the new primary spin-4 field [10] is

$$P_{4'}^{ww}(z) = \frac{2}{1221} \psi^a \partial^3 \psi^a(z) + \text{other lower order derivative terms} \tag{2.29}$$

whose number of terms is the same as the one for spin-4 field in (2.23). Note that these two currents do not appear in (2.8). The spin-4 field (2.29) will provide some component of superprimary field $\hat{O}_{\frac{7}{2}}(Z)$ or $\hat{O}_4(Z)$. Although the spin-4 field (2.22) or (2.28) is primary under the stress energy tensor, the OPE with spin-$\frac{3}{2}$ current has unusual behavior as mentioned before. If the primary field is one of the components in the superprimary field with given spin, it should transform like as (2.6) or (2.7). However, the OPE of this spin-4 field and the spin-$\frac{3}{2}$ current has third-order and fourth-order singular terms. This implies that there is no superpartner for this spin-4 field [8].

So far the spin-3 current and spin-$\frac{5}{2}$ current are given in (2.18) and (2.19) while the spin-$\frac{7}{2}$ field and the spin-$\frac{9}{2}$ field are found in (2.25) and (2.26) respectively. They are located in the first three supercurrents in the list (2.20). Now one should find out other higher spin currents, the superpartners of $O_{\frac{7}{2}}(z)$ and $O_{\frac{9}{2}}(z)$, in terms of eight fermion fields.

How to determine the other higher spin currents corresponding to $O_4(z)$ or $O_{4'}(z)$? Let us look at the OPE between the spin-$\frac{3}{2}$ current $G(z)$ and the spin-$\frac{7}{2}$ current $O_{\frac{7}{2}}(w)$. Since we have explicit forms in (2.17) and (2.25) respectively, we compute this OPE and it turns out that this OPE leads to the following first-order singular term with $\left(-\frac{1}{\sqrt{6}} P_{4'}^{uu} + \sqrt{6} P_{4'}^{ww}\right)(w)$ where the spin-4 currents are given in (2.23) and (2.29) as before. This implies that one can construct the following current of spin-4, due to the $\mathcal{N}=1$ supersymmetry,

$$O_4(z) = \left(-\frac{1}{\sqrt{6}} P_{4'}^{uu} + \sqrt{6} P_{4'}^{ww}\right)(z). \tag{2.30}$$

That is, one has

$$G(z)\, O_{\frac{7}{2}}(w) = \frac{1}{(z-w)}\, O_4(w) + \cdots. \tag{2.31}$$

One also checks that the OPE $G(z)$ with $O_4(w)$ leads to the expected singular terms with $O_{\frac{7}{2}}(w)$ by $\mathcal{N}=1$ supersymmetry. It turns out that

$$G(z)\, O_4(w) = \frac{1}{(z-w)^2}\, 7 O_{\frac{7}{2}}(w) + \frac{1}{(z-w)}\, \partial O_{\frac{7}{2}}(w) + \cdots. \tag{2.32}$$

Furthermore, one computes the OPE between the spin-$\frac{3}{2}$ current $G(z)$ and the spin-$\frac{9}{2}$ current $O_{\frac{9}{2}}(w)$ (2.26) in order to determine the superpartner with spin-4 field. It turns out,

[8] More explicitly, one obtains the OPE $G(z)\ (TT + \frac{42}{17} G\partial G - \frac{69}{34}\partial^2 T)(w) = -\frac{1}{(z-w)^4}\frac{1221}{68}\, G(w) - \frac{1}{(z-w)^3}\left(-\frac{1}{3}\right)\frac{1221}{68}\partial G(w) + \mathcal{O}((z-w)^{-2})$.

similar to (2.7), that

$$G(z)\, O_{\frac{9}{2}}(w) = \frac{1}{(z-w)^2}\, 8 O_{4'}(w) + \frac{1}{(z-w)}\, \partial O_{4'}(w) + \cdots, \tag{2.33}$$

where the superpartner of $O_{\frac{9}{2}}(z)$ is given by

$$O_{4'}(z) = \frac{1}{8}\left(\frac{16}{7}\sqrt{\frac{2}{3}} P_{4'}^{uu} - \frac{4}{7}\sqrt{6} P_{4'}^{ww}\right)(z). \tag{2.34}$$

In general, the coefficient in the second-order pole in (2.33) is equal to 2 times the spin of the current appearing in that singular term. In (2.33), $8 = 2 \times 4$ while in (2.32), $7 = 2 \times \frac{7}{2}$. Similarly, one has, like as (2.6),

$$G(z)\, O_{4'}(w) = \frac{1}{(z-w)}\, O_{\frac{9}{2}}(w) + \cdots. \tag{2.35}$$

Therefore, the supercurrents $\hat{O}_{\frac{7}{2}}(z)$ and $\hat{O}_4(z)$ in (2.20) are determined completely [9].

We have constructed the supercurrent $\hat{O}_{\frac{7}{2}}(Z)$ with (2.31) and (2.32) and $\hat{O}_4(Z)$ with (2.33) and (2.35) in the list of (2.20). Actually, these were found in [9]. Their component fields are given by (2.25), (2.30), (2.34) and (2.26). In $\mathcal{N}=1$ notation, the above superfusion rule between the supercurrent $\hat{W}(Z)$ and itself can be rewritten as $\left[\hat{W}\right]\left[\hat{W}\right] = \left[\hat{I}\right] + \left[\hat{O}_{\frac{7}{2}}\right] + \left[\hat{O}_4\right]$. The right hand side of this OPE contains two higher spin superprimary fields.

Then it is natural to consider the OPE between $\hat{W}(Z_1)$ and $\hat{O}_{\frac{7}{2}}(Z_2)$ which is the next lower higher spin supercurrent.

• The construction of higher spin supercurrents $\hat{O}_{4'}(Z)$ and $\hat{O}_{\frac{9}{2}}(Z)$

Now it is ready to obtain the other next lower higher spin currents in (2.20). Let us compute the OPE between the spin-$\frac{5}{2}$ current $U(z)$ given in (2.19) and the spin-$\frac{7}{2}$ current $O_{\frac{7}{2}}(w)$ given in (2.25). It turns out

$$\begin{aligned} U(z)\, O_{\frac{7}{2}}(w) &= \frac{1}{(z-w)^3}\, \frac{36}{185} W(w) \\ &+ \frac{1}{(z-w)^2}\left[(\frac{1}{3})\frac{36}{185}\partial W - \frac{6\sqrt{6}}{481}\left(GU - \frac{\sqrt{6}}{3}\partial W\right) + O_{4''}\right](w) \\ &+ \frac{1}{(z-w)}\left[(\frac{1}{14})\frac{36}{185}\partial^2 W - (\frac{3}{8})\frac{6\sqrt{6}}{481}\partial\left(GU - \frac{\sqrt{6}}{3}\partial W\right) + (\frac{3}{8})\partial O_{4''}\right. \\ &+ \left.\frac{764}{8325}\left(TW - \frac{3}{14}\partial^2 W\right) + \frac{287}{5550\sqrt{6}}\left(G\partial U - \frac{5}{3}\partial G U - \frac{\sqrt{6}}{7}\partial^2 W\right) + O_5\right](w) \\ &+ \cdots. \end{aligned} \tag{2.36}$$

[9]It is convenient to write the OPEs $G(z)\, P_{4'}^{uu}(w) = \frac{1}{(z-w)^2}\sqrt{6} O_{\frac{7}{2}}(w) + \frac{1}{(z-w)}\left(\frac{\sqrt{6}}{6}\partial O_{\frac{7}{2}} + 2\sqrt{6} O_{\frac{9}{2}}\right)(w) + \cdots$ and $G(z)\, P_{4'}^{ww}(w) = \frac{1}{(z-w)^2} 4\sqrt{\frac{2}{3}} O_{\frac{7}{2}}(w) + \frac{1}{(z-w)}\left(\frac{4}{7}\sqrt{6}\partial O_{\frac{7}{2}} + \sqrt{\frac{2}{3}} O_{\frac{9}{2}}\right)(w) + \cdots$. This is the reason why the primary fields $P_{4'}^{uu}(z)$ and $P_{4'}^{ww}(w)$ are not good for the $\mathcal{N}=1$ supercurrents.

For the primary field $W(w)$ with the structure constant $\frac{36}{185}$ in the right hand side, the relative coefficients for its descendant fields appearing in various singular terms can be read off from (1.3). See also the Appendix A for detailed coefficients in the structure constants. Let us look at the second-order singular terms. The first term originating from $W(w)$ is fixed. How one can see the next quasi-primary or primary field? From the second-order pole, one can compute the OPE between $T(z)$ and the second order-pole subtracted by $(\frac{1}{3})\frac{36}{185}\partial W(w)$, in order to extract the possible quasi-primary fields (i.e., the exact expressions and the number of quasi-primary fields). Then one obtains $T(z)\left(\{UO_{\frac{7}{2}}\}_{-2}-(\frac{1}{3})\frac{36}{185}\partial W\right)(w) = +\mathcal{O}((z-w)^{-2})$. In other words, it transforms as a primary field. However, one also computes $G(z)\left(\{UO_{\frac{7}{2}}\}_{-2}-(\frac{1}{3})\frac{36}{185}\partial W\right)(w) = \frac{1}{(z-w)^3}\,\frac{2\sqrt{6}}{37}\,U(w)+\mathcal{O}((z-w)^{-2})$ [10]. This implies that the remaining terms in the second-order pole contain a primary field with unusual behavior with $G(z)$. Can we obtain this explicitly? Yes, by subtracting $GU(w)$ plus other derivative term with unknown coefficient into above second-order singular terms, one can remove the unwanted third-order pole which is proportional to $U(w)$ by choosing the correct coefficient. This is due to the fact that when one computes the OPE $G(z)$ with $GU(w)$, one has $U(w)$ term in the third-order singular term via (2.2). Therefore, one can think of $(GU-\frac{\sqrt{6}}{4}\partial W)(w)$ as a possible candidate for the primary field we want to subtract [11]. The coefficient $-\frac{6\sqrt{6}}{481}$ in (2.36) in front of this field is fixed by requiring that there should be no third-order pole from the superprimary condition. Then we are left with the following primary field. The spin-4 primary field is given by

$$O_{4''}(z) = \frac{384}{13}\sqrt{\frac{2}{5}}\,i\,\psi^1\psi^2\psi^3\psi^4\psi^5\psi^6\psi^7\psi^8(z) + \text{other derivative terms.} \tag{2.37}$$

Let us consider the last first-order singular terms. The first line of these terms in (2.36) describes the descendant field for the spin-3 current and two descendant fields for the spin-4 primary fields. As before, we compute the difference between the whole first-order singular terms and those three descendant terms and make sure that there should exist two quasi-primary fields presented in the second line of the first order-singular terms in (2.36). The OPE $T(z)$ with $\left[\{UO_{\frac{7}{2}}\}_{-1}-\text{first line}\right](w)$ leads to $\frac{1}{(z-w)^4}\,\frac{162}{259}W(w)+\mathcal{O}((z-w)^{-2})$. This implies that one should consider the extra quasi-primary field, $TW(w)$ plus derivative terms, in order to cancel the fourth-order term $\frac{162}{259}W(w)$ for the superprimary condition. Furthermore, one should also compute the OPE between $G(z)$ and $\left[\{UO_{\frac{7}{2}}\}_{-1}-\text{first line}\right](w)$ and it leads to $\frac{1}{(z-w)^4}\,\frac{33}{518}\sqrt{\frac{3}{2}}\,U(w)+\frac{1}{(z-w)^3}\,(-\frac{1}{5})\frac{33}{518}\sqrt{\frac{3}{2}}\,\partial U(w)+\mathcal{O}((z-w)^{-2})$. This also indicates that

[10] We use a simplified notation here $\{UO_{\frac{7}{2}}\}_{-2}(w)$ for the second order pole of (2.36) in the spirit of [13, 3, 16].

[11] One obtains the OPE $G(z)\ (GU-\frac{\sqrt{6}}{4}\partial W)(w) = \frac{1}{(z-w)^3}\,\frac{13}{3}U(w)+\mathcal{O}((z-w)^{-2})$. See also the Appendix C. In $\mathcal{N}=1$ supercurrent, this primary field originates from $\hat{T}\hat{W}(Z_2)$ [9].

one should consider the extra quasi-primary field consisting of $G\partial U(w), \partial GU(w)$ and other derivative terms, in order to remove the higher singular terms above. Then, finally the consistent coefficients, $\frac{764}{8325}$ and $\frac{287}{5550\sqrt{6}}$, in the second line of first-order pole in (2.36) are fixed from the above analysis (i.e., superprimary condition) and we are left with the following spin-5 primary current

$$O_5(z) = -\frac{8}{225}\sqrt{\frac{2}{15}}\, i\, f^{abc} d^{ade}\, \psi^b \psi^c \psi^d \partial^3 \psi^e(z) + \text{other lower order derivative terms.} \quad (2.38)$$

Therefore, we have obtained two primary currents (2.37) and (2.38) where the former is θ-independent component field of $\hat{O}_{4'}(Z)$ and the latter is the θ-dependent component field of $\hat{O}_{\frac{9}{2}}(Z)$.

Let us compute the OPE of spin-$\frac{5}{2}$ current (2.19) and spin-4 current (2.30) in order to find the superpartners corresponding above two primary fields. One obtains each singular terms, starting from fourth-order singular term. We present the final result first and then explain how we obtain this result explicitly

$$\begin{aligned}
U(z)\, O_4(w) &= \frac{1}{(z-w)^4}\frac{6\sqrt{6}}{37}U(w) + \frac{1}{(z-w)^3}(\frac{1}{5})\frac{6\sqrt{6}}{37}\partial U(w) \\
&+\frac{1}{(z-w)^2}\left[(\frac{1}{30})\frac{6\sqrt{6}}{37}\partial^2 U + \frac{1596}{12025}\left(GW - \frac{1}{6\sqrt{6}}\partial^2 U\right) + \frac{232\sqrt{6}}{7215}\left(TU - \frac{1}{4}\partial^2 U\right) + P_{\frac{9}{2}}\right](w) \\
&+\frac{1}{(z-w)}\left[(\frac{1}{210})\frac{6\sqrt{6}}{37}\partial^3 U + (\frac{1}{3})\frac{1596}{12025}\partial\left(GW - \frac{1}{6\sqrt{6}}\partial^2 U\right) + (\frac{1}{3})\frac{232\sqrt{6}}{7215}\partial\left(TU - \frac{1}{4}\partial^2 U\right)\right. \\
&\left.+(\frac{1}{3})\partial P_{\frac{9}{2}} + Q_{\frac{11}{2}}\right](w) + \cdots. \quad (2.39)
\end{aligned}$$

The structure of the right hand side looks similar to (2.36) by changing $W(w)$ into $U(w)$ and vice versa. Once the structure constant $\frac{6\sqrt{6}}{37}$ in the highest order singular term is found, then the other relative coefficients in the lower singular terms, associated with the spin-$\frac{5}{2}$ current $U(w)$, are automatically determined through the formula (1.3). Now we should find out other terms in the right hand side of (2.39). Let us look at the nontrivial second-order singular terms. One should make sure that there exist three extra quasi-primary fields including the last primary field with the right structure constants. As we did before, we compute the OPE between $T(z)$ and $\left(\{UO_4\}_{-2} - (\frac{1}{30})\frac{6\sqrt{6}}{37}\partial^2 U\right)(w)$. This OPE has $\frac{18\sqrt{6}}{37}U(w)$ in the fourth-order singular term. This shows that the extra quasi-primary field should contain $TU(w)$ term. Moreover, one should also compute the OPE between $G(z)$ and $\left(\{UO_4\}_{-2} - (\frac{1}{30})\frac{6\sqrt{6}}{37}\partial^2 U\right)(w)$ and this OPE has the following third-order singular term $\frac{1}{(z-w)^3}\frac{276}{185}W(w)$. Then one can consider $GW(w)$ as a quasi-primary field with some derivative term. By subtracting these

two candidates from the second-order singular terms, one arrives at the final new spin-$\frac{9}{2}$ primary field

$$P_{\frac{9}{2}}(z) = \frac{1}{13}\sqrt{\frac{2}{5}}\, i\, d^{abc} f^{bde} f^{cfg} \psi^a \psi^d \psi^e \psi^f \partial^2 \psi^g(z) + \text{other first-order derivative terms}, \quad (2.40)$$

which will play the role of component field of some superprimary field.

Let us focus on the next first-order singular term. Now one can write down all possible descendant fields coming from the quasi-primary fields of spin-$\frac{5}{2}$ and of spin-$\frac{9}{2}$ with correct coefficient functions. Then we are left with the following quasi-primary field of spin-$\frac{11}{2}$

$$\begin{aligned} Q_{\frac{11}{2}}(z) &= -\frac{584}{7215}\sqrt{\frac{2}{3}}\left(T\partial U - \frac{5}{4}\partial T U - \frac{1}{7}\partial^3 U\right)(z) \\ &+ \frac{10}{481}\left(G\partial W - 2\partial G W - \frac{1}{21}\sqrt{\frac{2}{3}}\,\partial^3 U\right)(z) + \frac{3}{4}\left(G O_{4''} - \frac{2}{9}\partial O_{\frac{9}{2}'}\right)(z). \quad (2.41) \end{aligned}$$

Of course, this expression is written in terms of various WZW currents and it is nontrivial to write down this in terms of (2.41). Note that there exists a new primary field $O_{\frac{9}{2}'}(z)$ in the right hand side of (2.41). From the newly obtained spin-4 primary field in (2.37), one computes the OPE of $G(z)$ with $O_{4''}(w)$ and it leads to a nonzero first-order singular term which is equal to the above $O_{\frac{9}{2}'}(w)$ where

$$O_{\frac{9}{2}'}(z) = -\frac{8}{65}\sqrt{\frac{2}{5}}\, i\, d^{abc} f^{bde} f^{cfg} \psi^a \psi^d \psi^e \psi^f \partial^2 \psi^g(z) + \text{other first-order derivative terms}. (2.42)$$

For double check, one also computes the OPE between the $G(z)$ and this field $O_{\frac{9}{2}'}(w)$. This provides the consistent result. That is, the second-order singular term has $8 O_{4''}(w)$ while the first-order singular term is given by $\partial O_{4''}(w)$. Then these are combined into a single superprimary field $\hat{O}_{4'}(Z)$ as in (2.20). One should find out the remaining current $O_{\frac{9}{2}''}(z)$. The quasi-primary property of (2.41) can be checked by the following OPE with $T(z)$

$$\begin{aligned} T(z)\, Q_{\frac{11}{2}}(w) &= \frac{1}{(z-w)^5}\frac{72}{259}\sqrt{6}\, U(w) + \frac{1}{(z-w)^4}(-\frac{1}{5})\frac{72}{259}\sqrt{6}\,\partial U(w) \\ &+ \mathcal{O}((z-w)^{-2}). \end{aligned} \quad (2.43)$$

There is no third-order pole. Furthermore, one also has the following OPE with the spin-$\frac{3}{2}$ current

$$\begin{aligned} G(z)\, Q_{\frac{11}{2}}(w) &= \frac{1}{(z-w)^4}\frac{48}{1295}W(w) \\ &+ \frac{1}{(z-w)^3}\left[(-\frac{1}{6})\frac{48}{1295}\partial W + \frac{16}{3}O_{4''} + \frac{32}{481}\sqrt{6}\left(GU - \frac{\sqrt{6}}{3}\partial W\right)\right](w) \\ &+ \mathcal{O}((z-w)^{-2}). \end{aligned} \quad (2.44)$$

Note that in (2.41), one sees that one can check three quantities specified by the bracket are quasi-primary fields. In the Appendices B, C, we present some properties of the various quasi-primary fields we are describing in this paper. According to the definition of quasi-primary field introduced in the introduction, any linear combination of quasi-primary fields leads to another quasi-primary field which can be written in terms of the known currents.

Let us consider the next OPE between the spin-3 current (2.18) and the spin-$\frac{7}{2}$ current (2.25) in order to complete the OPE between $\hat{W}(Z_1)$ and $\hat{O}_{\frac{7}{2}}(Z_2)$. The result is as follows:

$$\begin{aligned} W(z)\, O_{\frac{7}{2}}(w) &= \frac{1}{(z-w)^4}\,\frac{6}{37}U(w) + \frac{1}{(z-w)^3}\,(\frac{2}{5})\frac{6}{37}\partial U(w) \\ &+ \frac{1}{(z-w)^2}\left[(\frac{1}{10})\frac{6}{37}\partial^2 U + \frac{412}{7215}\left(TU - \frac{1}{4}\partial^2 U\right) + \frac{116\sqrt{6}}{12025}\left(GW - \frac{1}{6\sqrt{6}}\partial^2 U\right) + \frac{1}{\sqrt{6}}P_{\frac{9}{2}} \right. \\ &+ \left. \frac{1}{\sqrt{6}}O_{\frac{9}{2}'}\right](w) + \frac{1}{(z-w)}\left[(\frac{2}{105})\frac{6}{37}\partial^3 U + (\frac{4}{9})\frac{412}{7215}\partial\left(TU - \frac{1}{4}\partial^2 U\right)\right. \\ &+ \left.(\frac{4}{9})\frac{116\sqrt{6}}{12025}\partial\left(GW - \frac{1}{6\sqrt{6}}\partial^2 U\right) + (\frac{4}{9})\frac{1}{\sqrt{6}}\partial P_{\frac{9}{2}} + (\frac{4}{9})\frac{1}{\sqrt{6}}\partial O_{\frac{9}{2}'} + Q_{\frac{11}{2}'}\right](w) + \cdots . \quad (2.45) \end{aligned}$$

This looks similar to the previous OPE (2.39). As before, once the structure constant appearing in front of $U(w)$ in the right hand side of (2.45) is found from the corresponding singular term with WZW currents, the relevant coefficients associated with its descendant fields in the second-order and first-order poles are known from (1.3). Then the next step is to look at the next nontrivial lower singular terms in order to see whether there exists a new primary field or not. If not, one should write down the singular terms in terms of known quasi-primary fields or new unknown quasi-primary fields. One has to compute the OPE between $T(z)$ and $\left(\{WO_{\frac{7}{2}}\}_{-2} - (\frac{1}{10})\frac{6}{37}\partial^2 U\right)(w)$ which becomes $-\frac{1}{(z-w)^4}\,\frac{9}{37}U(w) - \frac{1}{(z-w)^3}\,(\frac{3}{5})\frac{9}{37}\partial U(w) + \mathcal{O}((z-w)^{-2})$. On the other hand, the OPE with $G(z)$ leads to $-\frac{6\sqrt{6}}{185}W(w)$ in the third-order singular term. This indicates that one can extract the two quasi-primary fields in the second-order singular terms and the structure constants are fixed by primary condition. The remaining terms are characterized by two independent spin-$\frac{9}{2}$ currents $P_{\frac{9}{2}}(w)$ (2.40) and $O_{\frac{9}{2}'}(w)$ (2.42) we have considered before. Now we describe the first-order singular terms. Since we have found two quasi-primary fields as well as two spin-$\frac{9}{2}$ primary fields at the second-order singular terms, their coefficient functions are determined without any ambiguities. So we should look for any quasi-primary fields after extracting those known field contents from the first-order pole. Then the remaining field is spin-$\frac{11}{2}$ quasi-primary field

$$Q_{\frac{11}{2}'}(z) \;=\; -\frac{146}{7215}\sqrt{\frac{2}{3}}\left(G\partial W - 2\partial G W - \frac{1}{21}\sqrt{\frac{2}{3}}\partial^3 U\right)(z)$$

$$+ \frac{16}{64935}\left(T\partial U-\frac{5}{4}\partial TU-\frac{1}{7}\partial^3 U\right)(z)+\frac{1}{4}\sqrt{\frac{3}{2}}\left(GO_{4''}-\frac{2}{9}\partial O_{\frac{9}{2}'}\right)(z). \quad (2.46)$$

All the field contents for this quasi-primary field are given in terms of previously determined quasi-primary fields. Each quasi-primary field in (2.46) also appears in (2.41). The only difference is the relative coefficients between them. As we have seen in (2.43) and (2.44), one computes the following OPE

$$\begin{aligned} T(z)\, Q_{\frac{11}{2}'}(w) &= \frac{1}{(z-w)^5}\frac{80}{777}U(w)+\frac{1}{(z-w)^4}(-\frac{1}{5})\frac{80}{777}\partial U(w) \\ &+ \mathcal{O}((z-w)^{-2}). \end{aligned} \quad (2.47)$$

There is no third-order pole. Similarly, one has

$$\begin{aligned} G(z)\, Q_{\frac{11}{2}'}(w) &= \frac{1}{(z-w)^4}\frac{144}{1295}\sqrt{6}W(w) \\ &+ \frac{1}{(z-w)^3}\left[(-\frac{1}{6})\frac{144}{1295}\partial W+\frac{8}{3}\sqrt{\frac{2}{3}}O_{4''}+\frac{32}{481}\left(GU-\frac{\sqrt{6}}{3}\partial W\right)\right](w) \\ &+ \mathcal{O}((z-w)^{-2}). \end{aligned} \quad (2.48)$$

Of course, the field contents in the right hand side of (2.47) and (2.48) are the same as those in (2.43) and (2.44), as we expected. So far, we have found two primary fields (2.40) and (2.42) which will play the role of undetermined spin-$\frac{9}{2}$ current $O_{\frac{9}{2}''}(z)$.

We consider the OPE between spin-3 current (2.18) and spin-4 current (2.30). One obtains

$$\begin{aligned} & W(z)\, O_4(w) = \frac{1}{(z-w)^4}\frac{48\sqrt{6}}{185}W(w) \\ & +\frac{1}{(z-w)^3}\left[(\frac{1}{3})\frac{48\sqrt{6}}{185}\partial W+\sqrt{\frac{2}{3}}O_{4''}+\frac{12}{481}\left(GU-\frac{\sqrt{6}}{3}\partial W\right)\right](w) \\ & +\frac{1}{(z-w)^2}\left[(\frac{1}{14})\frac{48\sqrt{6}}{185}\partial^2 W+(\frac{3}{8})\sqrt{\frac{2}{3}}\partial O_{4''}+(\frac{3}{8})\frac{12}{481}\partial\left(GU-\frac{\sqrt{6}}{3}\partial W\right)\right. \\ & \left.+5\sqrt{\frac{2}{3}}O_5+Q_5\right](w) \\ & +\frac{1}{(z-w)}\left[(\frac{1}{84})\frac{48\sqrt{6}}{185}\partial^3 W+(\frac{1}{12})\sqrt{\frac{2}{3}}\partial^2 O_{4''}+(\frac{1}{12})\frac{12}{481}\partial^2\left(GU-\frac{\sqrt{6}}{3}\partial W\right)\right. \\ & \left.+(\frac{2}{5})5\sqrt{\frac{2}{3}}\partial O_5+(\frac{2}{5})\partial Q_5+Q_6\right](w). \end{aligned} \quad (2.49)$$

By identifying the highest singular term from the explicit expression of the OPE, one can write down the descendant fields with correct coefficient functions for the spin-3 current $W(w)$ in

the right hand side. Now the first nontrivial third-order term can be analyzed. One computes the OPE between $T(z)$ and $\left(\{W O_4\}_{-3} - (\frac{1}{3})\frac{48\sqrt{6}}{185}\partial W\right)(w)$ and this OPE does not produce any higher order (greater than 2) singular terms. This fact leads to the appearance of primary field. What about the OPE between $G(z)$ and $\left(\{W O_4\}_{-3} - (\frac{1}{3})\frac{48\sqrt{6}}{185}\partial W\right)(w)$? The third-order term of this OPE contains $\frac{4}{37}U(w)$. Therefore, one can extract the primary field containing $GU(w)$ from the third-order pole. We are left with the spin-4 current (2.37) we have described before. Since the third-order singular terms are determined completely, let us move on the second-order term. Due to the formula (1.3), one easily extracts the first line of the second-order terms in (2.49) from the whole second-order pole. Then one can compute the OPE with $T(z)$ and $(\{W O_4\}_{-2} - \text{first line})(w)$. The nontrivial part of this OPE contains $\frac{1248\sqrt{6}}{1295}W(w)$ at fourth-order pole. For the OPE between $G(z)$ and $(\{W O_4\}_{-2} - \text{first line})(w)$, there are $\frac{573}{518}U(w)$ in the fourth-order term and $(-\frac{1}{5})\frac{573}{518}\partial U(w)$ in the third-order term. These two facts lead us to subtract the two quasi-primary fields corresponding to $TW(w)$(and derivative term) and $G\partial U(w)$(and other terms) respectively. After subtracting these two quasi-primary fields properly in the second-order pole, we end up with the spin-5 primary field (2.38) we discussed before. Moreover, one has the following spin-5 quasi-primary field

$$Q_5(z) = \frac{458}{1665}\sqrt{\frac{2}{3}}\left(TW - \frac{3}{14}\partial^2 W\right)(z) - \frac{19}{3330}\left(G\partial U - \frac{5}{3}\partial G U - \frac{\sqrt{6}}{7}\partial^2 W\right)(z) \qquad (2.50)$$

and its OPEs with $T(z)$ and $G(z)$ are given by

$$\begin{aligned} T(z)\,Q_5(w) &= \frac{1}{(z-w)^4}\frac{1248\sqrt{6}}{1295}W(w) + \mathcal{O}((z-w)^{-2}), \\ G(z)\,Q_5(w) &= -\frac{1}{(z-w)^4}\frac{573}{2590}U(w) - \frac{1}{(z-w)^3}(-\frac{1}{5})\frac{573}{2590}\partial U(w) \\ &+ \mathcal{O}((z-w)^{-2}). \end{aligned} \qquad (2.51)$$

One sees that there is no third-order pole in the first equation of (2.51). All the structure of the second-order pole is fixed, and one also describes the last first-order pole. By subtracting the three descendant fields (coming from spin-3 primary field and two spin-4 quasi-primary fields) and the remaining two descendant fields coming from the spin-5 quasi-primary fields appearing in the second-order pole, one realizes that there exists a spin-6 quasi-primary field that is

$$\begin{aligned} Q_6(z) &= \frac{32}{12025}\left(G\partial^2 U - 4\partial G\partial U + \frac{5}{2}\partial^2 G U - \frac{1}{2\sqrt{6}}\partial^3 W\right)(z) \\ &- \frac{192}{12025}\sqrt{6}\left(T\partial W - \frac{3}{2}\partial T W - \frac{1}{8}\partial^3 W\right)(z) + \frac{1}{2}\sqrt{\frac{3}{2}}\left(T O_{4''} - \frac{1}{6}\partial^2 O_{4''}\right)(z) \\ &- \frac{1}{4}\sqrt{\frac{3}{2}}\left(G O_{\frac{9}{2}'} - \frac{1}{9}\partial^2 O_{4''}\right)(z). \end{aligned} \qquad (2.52)$$

One can easily compute the following OPEs

$$
\begin{aligned}
T(z)\, Q_6(w) &= \frac{1}{(z-w)^5}\,\frac{288}{925}\sqrt{6}\,W(w) \\
&+ \frac{1}{(z-w)^4}\left[(-\frac{1}{6})\frac{288}{925}\sqrt{6}\,\partial W + 4\sqrt{\frac{2}{3}}\,O_{4''} + \frac{48}{481}\left(GU - \frac{\sqrt{6}}{3}\partial W\right)\right](w) \\
&+ \mathcal{O}((z-w)^{-2}), \\
G(z)\, Q_6(w) &= \frac{1}{(z-w)^5}\,\frac{64}{185}\,U(w) + \frac{1}{(z-w)^4}\,(-\frac{2}{5})\frac{64}{185}\partial U(w) \\
&+ \frac{1}{(z-w)^3}\left[(\frac{1}{30})\frac{64}{185}\partial^2 U + \frac{32}{481}\sqrt{6}\left(GW - \frac{1}{6\sqrt{6}}\partial^2 U\right)\right. \\
&- \left.\frac{64}{481}\left(TU - \frac{1}{4}\partial^2 U\right) - \frac{8}{3}\sqrt{\frac{2}{3}}\,O_{\frac{9}{2}'}\right](w) + \mathcal{O}((z-w)^{-2}).
\end{aligned} \tag{2.53}
$$

One sees that there is no third-order pole in the first equation of (2.53).

Therefore, we have constructed the supercurrent $\hat{O}_{4'}(Z)$ and $\hat{O}_{\frac{9}{2}}(Z)$ in the list of (2.20). Originally, the expression $\hat{O}_{4'}(Z)$ was in [9]. Their component fields are given by (2.37), (2.42) and (2.38). Furthermore, the remaining component field is written as

$$
O_{\frac{9}{2}''}(z) = O_{\frac{9}{2}'}(z) + \frac{8}{5}P_{\frac{9}{2}}(z), \tag{2.54}
$$

where the component fields are given in (2.42) and (2.40). In order to see whether (2.54) is the right superpartner of the current $O_5(z)$, one can check the OPE between $G(z)$ and $O_{\frac{9}{2}''}(w)$ and it turns out that the first-order pole provides the spin-5 exactly like as (2.31). Moreover, the OPE between $G(z)$ and $O_5(w)$ gives the correct singular terms where the second order pole has $9O_{\frac{9}{2}''}(w)$ and the first-order pole has $\partial O_{\frac{9}{2}''}(w)$ like as (2.32) [12]. One has the following superfusion rule $\left[\hat{W}\right]\left[\hat{O}_{\frac{7}{2}}\right] = \left[\hat{W}\right] + \left[\hat{O}_{4'}\right] + \left[\hat{O}_{\frac{9}{2}}\right]$. Now it is ready to compute the OPE between $\hat{W}(Z_1)$ and $\hat{O}_{4'}(Z_2)$ where we will find the remaining higher spin currents.

- The construction of $\hat{O}_{\frac{11}{2}}(Z)$ and $\hat{O}_6(Z)$

Let us describe the OPE between the spin-$\frac{5}{2}$ current (2.19) and the spin-4 current (2.37). We present the final result.

$$
\begin{aligned}
U(z)\, O_{4''}(w) &= \frac{1}{(z-w)^3}\,\frac{888}{65}O_{\frac{7}{2}}(w) + \frac{1}{(z-w)^2}\left[(\frac{2}{7})\frac{888}{65}\partial O_{\frac{7}{2}} - \frac{88}{65}O_{\frac{9}{2}}\right](w) \\
&+ \frac{1}{(z-w)}\left[(\frac{3}{56})\frac{888}{65}\partial^2 O_{\frac{7}{2}} - (\frac{1}{3})\frac{88}{65}\partial O_{\frac{9}{2}} + \frac{288}{61}\left(TO_{\frac{7}{2}} - \frac{3}{16}\partial^2 O_{\frac{7}{2}}\right)\right.
\end{aligned}
$$

[12] It is convenient to obtain the OPE $G(z)\ P_{\frac{9}{2}}(w) = -\frac{1}{(z-w)^2}5O_{4''}(w) + \frac{1}{(z-w)}\frac{5}{8}\left(O_5 - \partial O_{4''}\right)(w) + \cdots$. Therefore, one should consider the equation (2.54) in order to remove the unwanted terms $O_{4''}(w)$ and its descendant field.

$$+\frac{58\sqrt{6}}{793}\left(GP^{uu}_{4'}-\frac{4\sqrt{6}}{9}\partial O_{\frac{9}{2}}-\frac{\sqrt{6}}{56}\partial^2 O_{\frac{7}{2}}\right)$$
$$-\frac{5493}{3965}\sqrt{\frac{3}{2}}\left(GP^{ww}_{4'}-\frac{2}{9}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}}-\frac{1}{14}\sqrt{\frac{2}{3}}\partial^2 O_{\frac{7}{2}}\right)+O_{\frac{11}{2}}\Bigg](w)+\cdots. \quad (2.55)$$

From the highest singular term in (2.55), the structure constant for spin-$\frac{7}{2}$ current can be determined and the formula (1.3) with this numerical value gives the explicit form for some part of the next singular terms. As before, one can extract this descendant field in the second-order singular term and the remaining term can be written in terms of the spin-$\frac{9}{2}$ current (2.26) we have seen before. Now one can go further. In order to find out the nontrivial first-order singular term completely, one can compute the OPE between the $T(z)$ and $(\{UO_{4''}\}_{-1}-(\frac{3}{56})\frac{888}{65}\partial^2 O_{\frac{7}{2}}+(\frac{1}{3})\frac{88}{65}\partial O_{\frac{9}{2}})(w)$. This leads to the nontrivial fourth-order pole with $\frac{2997}{65}O_{\frac{7}{2}}(w)$ and one can extract the quasi-primary field containing $TO_{\frac{7}{2}}$ from the first-order pole. Furthermore, the OPE between $G(z)$ and $(\{UO_{4''}\}_{-1}-(\frac{3}{56})\frac{888}{65}\partial^2 O_{\frac{7}{2}}+(\frac{1}{3})\frac{88}{65}\partial O_{\frac{9}{2}})(w)$ has the nontrivial third-order singular term with $\left(-\frac{439}{195}\sqrt{\frac{2}{3}}P^{uu}_{4'}+\frac{526}{65}\sqrt{\frac{2}{3}}P^{ww}_{4'}\right)(w)$. Then one can consider the two quasi-primary fields containing $GP^{uu}_{4'}(w)$ and $GP^{ww}_{4'}(w)$ respectively. Finally, one arrives at the new spin-$\frac{11}{2}$ current, by rearranging the first-order terms as done before,

$$O_{\frac{11}{2}}(z)=\frac{3}{1586\sqrt{2}}\,f^{abc}\psi^a\psi^b\partial^4\psi^c(z)+\text{other lower order derivative terms}, \quad (2.56)$$

which is the θ-independent term of $\hat{O}_{\frac{11}{2}}(Z)$.

Now let us consider the spin-$\frac{5}{2}$ current (2.19) and the spin-$\frac{9}{2}$ current (2.42), in order to determine other unknown higher spin currents. The result is

$$U(z)\;O_{\frac{9}{2}'}(w)=\frac{1}{(z-w)^3}\left[\frac{116}{65}\sqrt{\frac{2}{3}}P^{uu}_{4'}-\frac{584\sqrt{6}}{65}P^{ww}_{4'}\right](w)$$
$$+\frac{1}{(z-w)^2}\Bigg[(\frac{1}{4})\frac{116}{65}\sqrt{\frac{2}{3}}\partial P^{uu}_{4'}-(\frac{1}{4})\frac{584\sqrt{6}}{65}\partial P^{ww}_{4'}$$
$$+\frac{792}{65}\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\partial P^{ww}_{4'}\right)\Bigg](w)$$
$$+\frac{1}{(z-w)}\Bigg[(\frac{1}{24})\frac{116}{65}\sqrt{\frac{2}{3}}\partial^2 P^{uu}_{4'}-(\frac{1}{24})\frac{584\sqrt{6}}{65}\partial^2 P^{ww}_{4'}$$
$$+(\frac{3}{10})\frac{792}{65}\partial\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\partial P^{ww}_{4'}\right)$$
$$+\frac{41524\sqrt{6}}{91195}\left(TP^{uu}_{4'}-\frac{1}{6}\partial^2 P^{uu}_{4'}\right)-\frac{328993\sqrt{6}}{91195}\left(TP^{ww}_{4'}-\frac{1}{6}\partial^2 P^{ww}_{4'}\right)$$

$$
-\frac{43068}{19825}\left(G\partial O_{\frac{7}{2}}-\frac{7}{3}\partial G O_{\frac{7}{2}}+\frac{1}{9\sqrt{6}}\partial^2 P^{uu}_{4'}-\frac{1}{3}\sqrt{\frac{2}{3}}\partial^2 P^{ww}_{4'}\right)
$$
$$
-\frac{231}{299}\left(G O_{\frac{9}{2}}-\frac{2}{63}\sqrt{\frac{2}{3}}\partial^2 P^{uu}_{4'}+\frac{1}{21\sqrt{6}}\partial^2 P^{ww}_{4'}\right)+P_6\Bigg](w)+\cdots. \qquad (2.57)
$$

In this case, the first nontrivial primary fields in the right hand side are given by (2.23) and (2.29). How to extract the quasi-primary field in the second-order pole? Actually, there exists a primary field of spin-5 but its OPE with $G(z)$ has unusual behavior. See the Appendices B, C. There is third-order pole which is given by $\frac{4884}{65}O_{\frac{7}{2}}(w)$. Then one tries to write down the primary field as the one in the third term of second-order pole in (2.57). By taking the derivative to these second-order terms with appropriate coefficients coming from (1.3) and subtracting them, we are left with nontrivial terms in the first-order singular terms. We want to check whether there exist four quasi-primary fields and a single primary field right or not. In order to do this job, one should compute the OPE between $T(z)$ and $(\{UO_{\frac{9}{2}'}\}_{-1}-$first three descendant fields$)(w)$. This turns out that the fourth-order term of this OPE is given by $\frac{406}{65}\sqrt{\frac{2}{3}}P^{uu}_{4'}(w)-\frac{2044\sqrt{6}}{65}P^{ww}_{4'}(w)$. It is natural to consider the quasi-primary field containing $TP^{uu}_{4'}(w)$ and the quasi-primary field $TP^{ww}_{4'}(w)$ with possible derivative terms. Moreover, the OPE $G(z)$ with $(\{UO_{\frac{9}{2}'}\}_{-1}-$first three descendant fields$)(w)$ produces $-\frac{3108}{325}O_{\frac{7}{2}}(w)$ at the fourth-order term and $(-\frac{1}{7})\frac{3108}{325}\partial O_{\frac{7}{2}}(w)-\frac{2464}{195}O_{\frac{9}{2}}(w)$ at the third-order pole. This allows us to take the two additional quasi-primary fields as in (2.57). Finally, after subtracting the above four quasi-primary fields properly (with correct coefficients), we are left with the following new spin-6 primary field

$$
P_6(z)=\frac{17}{8418\sqrt{6}}\,\psi^a\partial^5\psi^a(z)+\text{other lower order derivative terms}. \qquad (2.58)
$$

This will play the role of the final spin-6 current we are looking for. So far, we have found two primary fields (2.56) and (2.58).

Now we continue to describe the OPE between the spin-3 current (2.18) and the spin-4 (2.37) current as follows:

$$
W(z)\,O_{4''}(w)=\frac{1}{(z-w)^3}\left[-\frac{328}{195}P^{uu}_{4'}+\frac{304}{65}P^{ww}_{4'}\right](w)
$$
$$
+\frac{1}{(z-w)^2}\left[-(\frac{3}{8})\frac{328}{195}\partial P^{uu}_{4'}+(\frac{3}{8})\frac{304}{65}\partial P^{ww}_{4'}+\frac{132\sqrt{6}}{65}\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\partial P^{ww}_{4'}\right)\right](w)
$$
$$
+\frac{1}{(z-w)}\Bigg[-(\frac{1}{12})\frac{328}{195}\partial^2 P^{uu}_{4'}+(\frac{1}{12})\frac{304}{65}\partial^2 P^{ww}_{4'}+(\frac{2}{5})\frac{132\sqrt{6}}{65}\partial\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\partial P^{ww}_{4'}\right)
$$
$$
-\frac{16896}{91195}\left(TP^{uu}_{4'}-\frac{1}{6}\partial^2 P^{uu}_{4'}\right)-\frac{24772}{91195}\left(TP^{ww}_{4'}-\frac{1}{6}\partial^2 P^{ww}_{4'}\right)
$$

$$
-\frac{4736\sqrt{6}}{10675}\left(G\partial O_{\frac{7}{2}}-\frac{7}{3}\partial G O_{\frac{7}{2}}+\frac{1}{9\sqrt{6}}\partial^2 P_{4'}^{uu}-\frac{1}{3}\sqrt{\frac{2}{3}}\partial^2 P_{4'}^{ww}\right)
$$

$$
-\frac{66\sqrt{6}}{1495}\left(G O_{\frac{9}{2}}-\frac{2}{63}\sqrt{\frac{2}{3}}\partial^2 P_{4'}^{uu}+\frac{1}{21\sqrt{6}}\partial^2 P_{4'}^{ww}\right)+P_{6'}\Bigg](w)+\cdots. \tag{2.59}
$$

The right hand side looks similar to the OPE (2.57). For given structure constants on the two spin-4 fields, its descendant field terms are completely fixed and one computes the OPE between $T(z)$ with $(\{WO_{4''}\}_{-2}-$two descendant fields$)(w)$ and this leads to the fact that there is no higher order singular term (the order is greater than 2) implying that the extra terms should correspond to the primary field. On the other hand, the OPE between $G(z)$ and $(\{WO_{4''}\}_{-2}-$ two descendant fields$)(w)$ provides the nontrivial third-order pole with $\frac{814\sqrt{6}}{65}O_{\frac{7}{2}}(w)$. Then the corresponding primary field can be written in terms of $GO_{\frac{7}{2}}(w)$ plus other derivative terms in (2.59). Based on the results for the second-order pole, one obtains the correct three derivative terms in the first-order singular terms (coming from the second-order pole). Then one can compute the OPE between $T(z)$ and $(\{WO_{4''}\}_{-1}-$ three descendant fields$)(w)$ which will determine the possible quasi-primary fields we should consider. There exists fourth-order pole in this OPE which is given by $-\frac{1312}{195}P_{4'}^{uu}(w)+\frac{1216}{65}P_{4'}^{ww}(w)$. This indicates that the quasi-primary field should contain $TP_{4'}^{uu}(w)$ and $TP_{4'}^{ww}(w)$ respectively. For the OPE $G(z)$ $(\{WO_{4''}\}_{-1}-$ three descendant fields$)(w)$, one sees that there are $\frac{3256\sqrt{6}}{325}O_{\frac{7}{2}}(w)$ at the fourth-order pole and $(-\frac{2}{7})\frac{3256\sqrt{6}}{325}\partial O_{\frac{7}{2}}(w)$ at the third-order pole. Finally, after extracting the new four quasi-primary fields from the first-order pole, one arrives at the following new spin-6 primary field

$$
P_{6'}(z)=\frac{121}{820755}\,\psi^a\partial^5\psi^a(z)+\text{other lower order derivative terms}. \tag{2.60}
$$

This is another candidate for the spin-6 current in the list (2.20).

Since we have found the spin-$\frac{11}{2}$ current (2.56), it is obvious to find out its superpartner $O_6(z)$ current. By computing the OPE $G(z)$ with $O_{\frac{11}{2}}(w)$ which should generate $O_6(w)$, one concludes that it consists of the linear combination of previous spin-6 fields (2.58) and (2.60)

$$
O_6(z)=-P_6(z)+\sqrt{6}P_{6'}(z). \tag{2.61}
$$

Similarly, one can compute the OPE between $G(z)$ and the current $O_6(w)$ and it turns out that the second order pole has $11O_{\frac{11}{2}}(w)$ and the first-order pole has $\partial O_{\frac{11}{2}}(w)$, as expected. Therefore, we are left with two unknown spin-$\frac{13}{2}$ current and its superpartner spin-6 current. However, the last spin-6 current can be obtained from the previous independent spin-6 currents. Effectively, one is left with the highest spin-$\frac{13}{2}$ current in (2.20).

Let us consider the last most complicated OPE between the spin-3 current (2.18) and the spin-$\frac{9}{2}$ current (2.42) in order to find out the last unknown spin-$\frac{13}{2}$ current:

$$
\begin{aligned}
W(z)\, O_{\frac{9}{2}'}(w) &= \frac{1}{(z-w)^4}\frac{444\sqrt{6}}{65}O_{\frac{7}{2}} + \frac{1}{(z-w)^3}\left[(\frac{2}{7})\frac{444\sqrt{6}}{65}\partial O_{\frac{7}{2}} - \frac{88}{13}\sqrt{\frac{2}{3}}O_{\frac{9}{2}}\right](w) \\
&+ \frac{1}{(z-w)^2}\left[(\frac{3}{56})\frac{444\sqrt{6}}{65}\partial^2 O_{\frac{7}{2}} - (\frac{1}{3})\frac{88}{13}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}} + \frac{1}{\sqrt{6}}O_{\frac{11}{2}} + \frac{19224\sqrt{6}}{3965}\left(TO_{\frac{7}{2}} - \frac{3}{16}\partial^2 O_{\frac{7}{2}}\right)\right. \\
&\left. + \frac{8342}{3965}\left(GP^{uu}_{4'} - \frac{4\sqrt{6}}{9}\partial O_{\frac{9}{2}} - \frac{\sqrt{6}}{56}\partial^2 O_{\frac{7}{2}}\right) - \frac{102117}{7930}\left(GP^{ww}_{4'} - \frac{2}{9}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}} - \frac{1}{14}\sqrt{\frac{2}{3}}\partial^2 O_{\frac{7}{2}}\right)\right](w) \\
&+ \frac{1}{(z-w)}\left[(\frac{1}{126})\frac{444\sqrt{6}}{65}\partial^3 O_{\frac{7}{2}} - (\frac{1}{15})\frac{88}{13}\sqrt{\frac{2}{3}}\partial^2 O_{\frac{9}{2}} + (\frac{4}{11})\frac{1}{\sqrt{6}}\partial O_{\frac{11}{2}}\right. \\
&+ (\frac{4}{11})\frac{19224\sqrt{6}}{3965}\partial\left(TO_{\frac{7}{2}} - \frac{3}{16}\partial^2 O_{\frac{7}{2}}\right) + (\frac{4}{11})\frac{8342}{3965}\partial\left(GP^{uu}_{4'} - \frac{4\sqrt{6}}{9}\partial O_{\frac{9}{2}} - \frac{\sqrt{6}}{56}\partial^2 O_{\frac{7}{2}}\right) \\
&- (\frac{4}{11})\frac{102117}{7930}\partial\left(GP^{ww}_{4'} - \frac{2}{9}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}} - (\frac{1}{14})\sqrt{\frac{2}{3}}\partial^2 O_{\frac{7}{2}}\right) \\
&- \frac{2464}{1495}\sqrt{\frac{2}{3}}\left(TO_{\frac{9}{2}} - \frac{3}{20}\partial^2 O_{\frac{9}{2}}\right) - \frac{2368}{715}\sqrt{\frac{2}{3}}\left(T\partial O_{\frac{7}{2}} - \frac{7}{4}\partial T O_{\frac{7}{2}} - \frac{1}{9}\partial^3 O_{\frac{7}{2}}\right) \\
&- \frac{6104}{16445}\left(G\partial P^{uu}_{4'} - \frac{8}{3}\partial G P^{uu}_{4'} - \frac{\sqrt{6}}{5}\partial^2 O_{\frac{9}{2}} - \frac{2}{63}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right) \\
&\left. + \frac{8064}{3289}\left(G\partial P^{ww}_{4'} - \frac{8}{3}\partial G P^{ww}_{4'} - \frac{1}{5\sqrt{6}}\partial^2 O_{\frac{9}{2}} - \frac{8}{189}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right) + \frac{2}{11}\sqrt{\frac{2}{3}}O_{\frac{13}{2}}\right](w) \\
&+ \cdots . \qquad (2.62)
\end{aligned}
$$

The independent fields in the right hand side up to the second-order singular terms are exactly the same as the one in the OPE (2.55). First of all, one can easily check that the third-order pole in (2.62) contains a primary field of spin-$\frac{9}{2}$ current (2.26) after subtracting the descendant field for $O_{\frac{7}{2}}(w)$. For the second-order pole, one should find out new structures after subtracting the right quasi-primary fields with correct coefficients. Let us compute the OPE between $T(z)$ and $(\{WO_{\frac{9}{2}'}\}_{-2} - (\frac{3}{56})\frac{444\sqrt{6}}{65}\partial^2 O_{\frac{7}{2}} + (\frac{1}{3})\frac{88}{13}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}})(w)$. This OPE contains $\frac{3441}{65}\sqrt{\frac{3}{2}}O_{\frac{7}{2}}(w)$ at the fourth-order pole. The OPE between $G(z)$ and $(\{WO_{\frac{9}{2}'}\}_{-2} - (\frac{3}{56})\frac{444\sqrt{6}}{65}\partial^2 O_{\frac{7}{2}} + (\frac{1}{3})\frac{88}{13}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}})(w)$ produces $\frac{6887}{585}P^{uu}_{4'}(w) - \frac{14126}{195}P^{ww}_{4'}(w)$ at third-order pole. Then the correct three quasi-primary fields with the right coefficients can be subtracted from the second-order pole. After that, we are left with the spin-$\frac{11}{2}$ current (2.56). It is ready to find out the complete structure of the first-order singular terms. The OPE between $T(z)$ and $(\{WO_{\frac{9}{2}'}\}_{-1} -$ derivative terms$)(w)$ contains $\frac{8288\sqrt{6}}{715}O_{\frac{7}{2}}(w)$ at the fifth-order pole and

$(-\frac{1}{7})\frac{8288\sqrt{6}}{715}\partial O_{\frac{7}{2}}(w) - \frac{616\sqrt{6}}{65}O_{\frac{9}{2}}(w)$ at the fourth-order pole. Similarly, one has $-\frac{896}{715}P_{4'}^{uu}(w) - \frac{24192}{715}P_{4'}^{ww}(w)$ at the fourth-order pole and $-(-\frac{1}{8})\frac{896}{715}\partial P_{4'}^{uu}(w) - (-\frac{1}{8})\frac{24192}{715}\partial P_{4'}^{ww}(w)$ at the third-order pole when one computes the OPE $G(z)$ with $(\{WO_{\frac{9}{2}'}\}_{-1} -$ derivative terms$)(w)$. From this analysis, one can write down the possible four quasi-primary fields with fixed coefficients. In the Appendices B, C, one sees the various properties between the currents $T(z)$ or $G(z)$ and those quasi-primary fields. We will also explain the general structures of quasi-primary fields for given known quasi-primary fields, $\Phi_i(z)$ and $\Phi_j(z)$ found by [15]. The nice thing is that once the OPE $\Phi_i(z)\ \Phi_j(w)$ is found, then the quasi-primary fields containing the derivatives of these two quasi-primary fields are determined completely. By subtracting above terms in the first-order singular terms, one arrives at the following spin-$\frac{13}{2}$ primary current, the highest spin current in the list of (2.20).

$$O_{\frac{13}{2}}(z) = \frac{16\sqrt{2}}{4485}\, f^{abc}\psi^a\psi^b\partial^5\psi^c(z) + \text{other lower order derivative terms.} \tag{2.63}$$

In order to find out its superpartner, one should compute the OPE $G(z)$ with the spin-$\frac{13}{2}$ current (2.63). Finally, one finds its correct superpartner is

$$O_{6'}(z) = P_6(z) + \frac{7}{2}\sqrt{\frac{3}{2}}P_{6'}(z). \tag{2.64}$$

Furthermore, the OPE between $G(z)$ and $O_{6'}(w)$ leads to the first-order pole $O_{\frac{13}{2}}(w)$ as expected [13]. Therefore, we have constructed the supercurrent $\hat{O}_{\frac{11}{2}}(Z)$ and $\hat{O}_6(Z)$ in the list of (2.20). The former consists of (2.56) and (2.61) and the latter is given by (2.64) and (2.63). The $\mathcal{N}=1$ superfusion rule is given by $\left[\hat{W}\right]\left[\hat{O}_{4'}\right] = \left[\hat{O}_{\frac{7}{2}}\right] + \left[\hat{O}_4\right] + \left[\hat{O}_{\frac{11}{2}}\right] + \left[\hat{O}_6\right]$.

In summary, the OPEs we have described so far are given by (2.21), (2.24), (2.27), (2.36), (2.39), (2.45), (2.49), (2.55), (2.57), (2.59), and (2.62). In principle, one can do all the other OPEs one did not compute in this paper. During these computations, the 12 higher spin primary currents where the spins are greater than 3 we are looking for are found. Moreover, there exist four quasi-primary fields and a primary field but these can be obtained from the known primary currents $T(z), G(z), U(z), W(z), O_{4''}(z)$ or $O_{\frac{9}{2}'}(z)$. It would be interesting to write down the OPEs we have found in $\mathcal{N}=1$ superspace. All these computations are based on the $c=4$ eight free fermion model. In next section, we generalize this model to $c<4$ coset model.

[13] Of course, the OPEs between $G(z)$ and $P_6(w)$ (or $P_{6'}(w)$) can be obtained from the standard results for the OPEs between $G(z)$ and $O_6(w)$ (and $O_{6'}(w)$) with (2.61) and (2.64).

3 The higher spin currents in the $\mathcal{N}=1$ supersymmetric coset minimal model (1.2)

Let us consider the perturbations of the $k \rightarrow \infty$ model described in previous subsection. There are two spin-1 currents, $J^a(z)$ and $K^a(z)$ of level 3 and k which generate the algebra $(A_2^{(1)} \oplus A_2^{(1)})$. The OPE between the spin-1 currents $J^a(z)$ is given in (2.14) and the corresponding OPE for the spin-1 current $K^a(z)$ is

$$K^a(z)\, K^b(w) = -\frac{1}{(z-w)^2}\frac{k}{2}\delta^{ab} + \frac{1}{(z-w)} f^{abc} K^c(w) + \cdots. \tag{3.1}$$

The diagonal subalgebra $A_2^{(1)}$ generates the spin-1 current $J^a(z) + K^a(z)$ with level $k+3$.

The coset Virasoro algebra is generated by the following Sugawara stress energy tensor

$$T(z) = -\frac{1}{6} J^a J^a(z) - \frac{1}{(k+3)} K^a K^a(z) + \frac{1}{(k+6)} (J^a + K^a)(J^a + K^a)(z), \tag{3.2}$$

which commutes with the above spin-1 current $J^a(z) + K^a(z)$, as expected. As the $k \rightarrow \infty$ limit, the above (3.2) becomes (2.16). The OPE of this spin-2 current and itself is given by (2.3), by using the OPEs (2.14) and (3.1), where the coset central charge is characterized by the following function of k

$$c = 4\left[1 - \frac{18}{(k+3)(k+6)}\right], \qquad k = 1, 2, \cdots. \tag{3.3}$$

When $k=1$, this central charge reduces to the one in mininal extension given in the subsection 2.2.

By requiring that the spin-$\frac{3}{2}$ current should commute with the diagonal spin-1 current and should transform as primary field under the stress energy tensor (3.2), one determines as [9]

$$G(z) = -\frac{2k}{3\sqrt{3(k+3)(k+6)}} \psi^a \left(J^a - \frac{9}{k} K^a \right)(z), \tag{3.4}$$

which satisfies (2.2) with the central charge (3.3). Also this current reduces to (2.17) as the k goes to ∞.

Similarly, the higher spin-3 current can be fixed by above regularity condition with the diagonal spin-1 current and primary condition with the stress energy tensor $T(z)$ as well as the fact that the highest singular term should behave as $\frac{c}{3}$. Therefore, it reads as [9]

$$\begin{aligned} W(z) \;\; = \;\; & \frac{2i}{9(k+3)(k+6)\sqrt{30(2k+3)(2k+15)}} d^{abc} \Big[k(k+3)(2k+3) J^a J^b J^c \\ & -18(k+3)(2k+3) J^a J^b K^c + 162(k+3) J^a K^b K^c - 162 K^a K^b K^c \Big](z), \end{aligned} \tag{3.5}$$

which reduces to the previous expression (2.18) for $k \to \infty$ limit. As we did before, its fermionic superpartner can be obtained from the spin-$\frac{3}{2}$ current (3.4) and it leads to [9]

$$\begin{aligned} U(z) &= \frac{2\sqrt{6}i}{15\sqrt{10(k+3)(k+6)(2k+3)(2k+15)}} d^{abc} \Big[k(2k+3)\psi^a J^b J^c \\ &- 15(2k+3)\psi^a J^b K^c + 90\psi^a K^b K^c \Big](z), \end{aligned} \tag{3.6}$$

which also becomes (2.19) for $k \to \infty$ limit.

Now it is ready to construct the 12 higher spin currents in (2.20) for $c < 4$ coset model.

• The construction of $\hat{O}_{\frac{7}{2}}(Z)$ and $\hat{O}_4(Z)$

Now it is ready to compute the OPE between the spin-$\frac{5}{2}$ current (3.6) and itself. The only difference between the $c = 4$ model and the $c < 4$ model occurs in the k-dependence in front of (3.6) and there exists an extra current $K(z)$ dependence. Therefore, one expects that the computations are more involved. It turns out that

$$\begin{aligned} U(z)\, U(w) &= \frac{1}{(z-w)^5} \frac{8k(9+k)}{5(3+k)(6+k)} + \frac{1}{(z-w)^3} 2T(w) + \frac{1}{(z-w)^2} \partial T(w) \\ &+ \frac{1}{(z-w)} \left[\frac{3}{10}\partial^2 T + \frac{9(3+k)(6+k)}{2(66+63k+7k^2)} \left(T^2 - \frac{3}{10}\partial^2 T \right) + P_4^{uu} + P_{4'}^{uu} \right](w) \\ &+ \cdots. \end{aligned} \tag{3.7}$$

The algebraic structure is the same as (2.21) except that the k-dependence occurs in various places. The stress energy tensor is given by (3.2). The k-dependent primary spin-4, which is a generalization of (2.22), is given by [14]

$$\begin{aligned} P_4^{uu}(z) &= \frac{3(3+k)(6+k)(498+225k+25k^2)}{(-42+99k+11k^2)(378+333k+37k^2)} \\ &\times \left[-\frac{7}{10}\partial^2 T + \frac{17(3+k)(6+k)}{6(66+63k+7k^2)} \left(T^2 - \frac{3}{10}\partial^2 T \right) + G\partial G \right](z). \end{aligned} \tag{3.8}$$

Also the k-dependent generalization of (2.23) can be obtained

$$\begin{aligned} P_{4'}^{uu}(z) &= -\frac{2(-1+k)k(9+k)(18+17k+9k^2)}{(6+k)(-42+99k+11k^2)(378+333k+37k^2)} \psi^a \partial^3 \psi^a(z) \\ &+ \text{ other 1970 terms}. \end{aligned} \tag{3.9}$$

[14] One obtains the generalization of the footnote 8, i.e., the OPE $G(z)\, \left(-\frac{7}{10}\partial^2 T + \frac{17(3+k)(6+k)}{6(66+63k+7k^2)}\left(T^2 - \frac{3}{10}\partial^2 T\right) + G\partial G\right)(w) = -\frac{1}{(z-w)^4}\left[\frac{(-42+99k+11k^2)(378+333k+37k^2)}{8(3+k)(6+k)(66+63k+7k^2)}\right] G(w) - \frac{1}{(z-w)^3}\left(-\frac{1}{3}\right)\left[\frac{(-42+99k+11k^2)(378+333k+37k^2)}{8(3+k)(6+k)(66+63k+7k^2)}\right] \partial G(w) + \mathcal{O}((z-w)^{-2})$. One sees the fourth-order and third-order singular terms. Of course, there is an overall factor in (3.8) we did not consider in this computation.

Only $\psi^a(z)$ dependent terms have $(k-1)$ factor in (3.9). The presence of this current was noticed in [9] where there is no explicit form for this current. Note that the k-dependence in front of quasi-primary field occurs while there is no k-dependence in front of the stress energy tensor and its descendant fields in (3.7). Of course, the central term in the highest singular term of (3.7) is the usual expression $\frac{2}{5}c$ where c is given by (3.3). The result looks different from the original one given the equation (4.16) in [9] but these are the same by manipulating the singular terms appropriately. It is more obvious to see the quasi-primary field from (3.7) rather than from old one. Moreover, the equation (3.7) reduces to the equation (2.21) as $k \to \infty$. The OPE between $T(z)$ and the quasi-primary field $(T^2 - \frac{3}{10}\partial^2 T)(w)$ has nontrivial k-dependence and we present this OPE in the Appendix D. See also the Appendix E.

Now let us move the following OPE between the spin-3 current (3.5) and the spin-$\frac{5}{2}$ current (3.6)

$$\begin{aligned} W(z)\, U(w) &= \frac{1}{(z-w)^4}\frac{3}{\sqrt{6}}G(w) + \frac{1}{(z-w)^3}(\frac{2}{3})\frac{3}{\sqrt{6}}\partial G(w) \\ &+ \frac{1}{(z-w)^2}\left[(\frac{1}{4})\frac{3}{\sqrt{6}}\partial^2 G + \frac{11\sqrt{6}(3+k)(6+k)}{(378+333k+37k^2)}\left(GT - \frac{1}{8}\partial^2 G\right) + O_{\frac{7}{2}}\right](w) \\ &+ \frac{1}{(z-w)}\left[(\frac{1}{15})\frac{3}{\sqrt{6}}\partial^3 G + (\frac{4}{7})\frac{11\sqrt{6}(3+k)(6+k)}{(378+333k+37k^2)}\partial\left(GT - \frac{1}{8}\partial^2 G\right) + (\frac{4}{7})\partial O_{\frac{7}{2}}\right. \\ &+ \left.\frac{4\sqrt{6}(3+k)(6+k)}{7(-42+99k+11k^2)}\left(\frac{4}{3}T\partial G - G\partial T - \frac{4}{15}\partial^3 G\right) + O_{\frac{9}{2}}\right](w) + \cdots. \end{aligned} \tag{3.10}$$

As before, the k-dependence in front of quasi-primary field occurs. The equation (3.10) leads to the equation (2.24) as $k \to \infty$. The k dependent spin-$\frac{7}{2}$ (the corresponding $k \to \infty$ limit was given in (2.25)) has the following form

$$\begin{aligned} O_{\frac{7}{2}}(z) &= -\frac{\sqrt{2}(-1+k)k(9+k)(9+2k)}{(15+2k)\sqrt{18+9k+k^2}(378+333k+37k^2)}\, f^{abc}\,\psi^a\psi^b\partial^2\psi^c(z) \\ &+ \text{ other 746 terms,} \end{aligned} \tag{3.11}$$

where only $\psi^a(z)$-dependent terms have $(k-1)$ factor in (3.11). There are K^a-dependent terms and mixed terms between $\psi^a(z)$ and $K^a(z)$. Furthermore, the spin-$\frac{9}{2}$ current, which generalizes the previous expression (2.26), has

$$\begin{aligned} O_{\frac{9}{2}}(z) &= -\frac{8\sqrt{2}(-1+k)k(1+k)(9+k)}{21(15+2k)\sqrt{18+9k+k^2}(-42+99k+11k^2)}\, f^{abc}\,\psi^a\psi^b\partial^3\psi^c(z) \\ &+ \text{ other 3624 terms.} \end{aligned} \tag{3.12}$$

Also only $\psi^a(z)$ dependent terms have $(k-1)$ factor in (3.12). The OPEs between the $T(z)$ and the quasi-primary fields appearing in (3.10) contain the k-dependence and these OPEs

are given in the Appendix D. Similarly, the OPEs between the $G(z)$ and those quasi-primary fields are given in the Appendix E where one can find the k-dependence explicitly.

The generalization of (2.27) can be obtained and the spin-3 current OPE is given by

$$
\begin{aligned}
W(z)\, W(w) = & = \frac{1}{(z-w)^6}\frac{4k(9+k)}{3(3+k)(6+k)} + \frac{1}{(z-w)^4}\, 2T(w) + \frac{1}{(z-w)^3}\, \partial T(w) \\
& + \frac{1}{(z-w)^2}\left[(\frac{3}{10})\partial^2 T + \frac{16(3+k)(6+k)}{3(66+63k+7k^2)}\left(T^2 - \frac{3}{10}\partial^2 T\right) + P_4^{ww} + P_{4'}^{ww}\right](w) \\
& + \frac{1}{(z-w)}\left[(\frac{1}{15})\partial^3 T + (\frac{1}{2})\frac{16(3+k)(6+k)}{3(66+63k+7k^2)}\partial\left(T^2 - \frac{3}{10}\partial^2 T\right)\right. \\
& + \left.(\frac{1}{2})\partial P_4^{ww} + (\frac{1}{2})\partial P_{4'}^{ww}\right](w) + \cdots, \qquad (3.13)
\end{aligned}
$$

where the k-dependent generalization of (2.28), spin-4 primary field, has the following form

$$
\begin{aligned}
P_4^{ww}(z) & = -\frac{48(-1+k)(3+k)(6+k)(10+k)}{(-42+99k+11k^2)(378+333k+37k^2)} \\
& \times \left[-\frac{7}{10}\partial^2 T + \frac{17(3+k)(6+k)}{6(66+63k+7k^2)}\left(T^2 - \frac{3}{10}\partial^2 T\right) + G\partial G\right](z). \qquad (3.14)
\end{aligned}
$$

Due to the $(k-1)$ factor in (3.14), this current vanishes at the "minimal" extension we have described before. One obtains other spin-4 primary current which generalizes the equation (2.29)

$$
\begin{aligned}
P_{4'}^{ww}(z) & = \frac{4(-1+k)^2 k(9+k)(27+k)}{3(15+2k)(-42+99k+11k^2)(378+333k+37k^2)}\psi^a \partial^3 \psi^a(z) \\
& + \text{ other 1818 terms}, \qquad (3.15)
\end{aligned}
$$

where only $\psi^a(z)$ dependent terms have $(k-1)$ factor in (3.15). One easily sees the central term $\frac{c}{3}$ in (3.13).

Therefore, as for infinite k case, one finds the two primary currents (3.11) and (3.12). From the other two primary fields (3.9) and (3.15), one constructs the new primary currents (2.30) and (2.34). In other words, the four independent currents in (2.20) are found during the computations of the OPEs (3.7), (3.10) and (3.13). One can easily check that the OPEs between the spin-$\frac{3}{2}$ current and the above four independent currents are the same as the ones in (2.31), (2.32) , (2.33) and (2.35).

- The construction of $\hat{O}_{4'}(Z)$ and $\hat{O}_{\frac{9}{2}}(Z)$

Let us consider the OPE between the spin-$\frac{5}{2}$ current (3.6) and the spin-$\frac{7}{2}$ current (3.11)

$$
U(z)\, O_{\frac{7}{2}}(w) = \frac{1}{(z-w)^3}\, c_{uow}\, W(w)
$$

$$+\frac{1}{(z-w)^2}\left[\frac{1}{3}\,c_{uow}\,\partial W - c_{uogu}\,\left(GU - \frac{\sqrt{6}}{3}\partial W\right) + O_{4''}\right](w)$$
$$+\frac{1}{(z-w)}\left[\frac{1}{14}\,c_{uow}\,\partial^2 W - \frac{3}{8}\,c_{uogu}\,\partial\left(GU - \frac{\sqrt{6}}{3}\partial W\right) + \frac{3}{8}\partial O_{4''}\right.$$
$$\left. + \, c_{uotw}\,\left(TW - \frac{3}{14}\partial^2 W\right) + c_{uogu'}\,\left(G\partial U - \frac{5}{3}\partial G U - \frac{\sqrt{6}}{7}\partial^2 W\right) + O_5\right](w)$$
$$+\cdots. \tag{3.16}$$

Here the structure constants can be written as

$$\begin{aligned}
c_{uow} &= \frac{36(-1+k)(10+k)(9+2k)^2}{5(3+2k)(15+2k)(378+333k+37k^2)}, \\
c_{uogu} &= \frac{6\sqrt{6}(-1+k)(3+k)(6+k)(10+k)(9+2k)^2}{(3+2k)(15+2k)(90+117k+13k^2)(378+333k+37k^2)}, \\
c_{uotw} &= \frac{4(3+k)(6+k)(9+2k)^2(198+1719k+191k^2)}{45(3+2k)(15+2k)(74+45k+5k^2)(378+333k+37k^2)}, \\
c_{uogu'} &= \frac{(3+k)(6+k)(9+2k)^2(5562+2583k+287k^2)}{(30\sqrt{6}(3+2k)(15+2k)(74+45k+5k^2)(378+333k+37k^2)}.
\end{aligned} \tag{3.17}$$

Note that one uses the OPE between the current $G(z)$ and the spin-4 primary field in (3.16) appearing in the Appedix E in order to determine the complete coefficient functions in the right hand side of (3.16). Moreover, the OPEs between the spin-2 current $T(z)$ or $G(z)$ and the quasi-primary fields of spin 5 can be obtained from the Appendices D or E. There are $(k-1)$ factors in first two structure constants in (3.17). How do we obtain the coefficient c_{uogu} explicitly? We have seen the algebraic structure in (2.36) for infinite k. Since the coefficient c_{uow} is fixed from the third-order pole, the first term in the second-order pole is determined. Then by introducing the undetermined coefficient c_{uogu} in front of the primary field of spin-4, one can compute the OPE $G(z)$ with $\left(\{UO_{\frac{7}{2}}\}_{-2} - \frac{1}{3}\,c_{uow}\,\partial W + c_{uogu}\,(GU - \frac{\sqrt{6}}{3}\partial W)\right)(w)$ where c_{uow} is given in (3.17). Then the requirement that the third-order pole should vanish (i.e., primary condition) determines the constant c_{uogu} explicitly. Then we are left with the spin-4 primary field which is given by

$$\begin{aligned}
O_{4''}(z) &= \frac{384i\sqrt{\frac{2}{5}}(-1+k)k(1+k)(9+2k)\sqrt{45+36k+4k^2}}{(3+k)(3+2k)(15+2k)(90+117k+13k^2)}\psi^1\psi^2\psi^3\psi^4\psi^5\psi^6\psi^7\psi^8(z) \\
&+ \text{ other 1376 terms},
\end{aligned} \tag{3.18}$$

where only $\psi^a(z)$ dependent terms have $(k-1)$ factor in (3.18).

What happens for the next-order pole? Since one knows the algebraic structure completely except the k-dependent coefficient functions, one computes the OPEs between $T(z)(G(z))$

with $\Big(\{UO_{\frac{7}{2}}\}_{-1}$ – three descendant terms – two quasi-primary terms$\Big)(w)$ with undetermined coefficients c_{uotw} and $c_{uogu'}$. One expects that this should transform very specially. In other words, there should be no higher order terms where the order is greater than 2 (once again the primary condition). This enables us to fix the above two constants as in (3.17). Therefore, we are left with the following spin-5 primary current

$$\begin{aligned} O_5(z) &= -\frac{2i\sqrt{\frac{2}{15}}(-1+k)k(9+k)(9+2k)^2(9+4k)}{45(3+k)(6+k)(15+2k)\sqrt{45+36k+4k^2}(74+45k+5k^2)} f^{abc}d^{ade}\,\psi^b\psi^c\psi^d\partial^3\psi^e(z) \\ &+ \text{ other 9603 terms}, \end{aligned} \tag{3.19}$$

where only $\psi^a(z)$ dependent terms have $(k-1)$ factor in (3.19). The corresponding $k \to \infty$ limit expressions are (2.37) and (2.38) respectively.

One has the following OPE between the spin-$\frac{5}{2}$ current and the spin-4 current as follows:

$$\begin{aligned} & U(z)\, O_4(w) = \frac{1}{(z-w)^4}\, c_{uou}\, U(w) + \frac{1}{(z-w)^3}\frac{1}{5}\, c_{uou}\, \partial U(w) \\ &+ \frac{1}{(z-w)^2}\left[\frac{1}{30}\, c_{uou}\, \partial^2 U + c_{uogw}\left(GW - \frac{1}{6\sqrt{6}}\partial^2 U\right) + c_{uotu}\left(TU - \frac{1}{4}\partial^2 U\right) + P_{\frac{9}{2}}\right](w) \\ &+ \frac{1}{(z-w)}\left[\frac{1}{210}\, c_{uou}\, \partial^3 U + \frac{1}{3}\, c_{uogw}\, \partial\left(GW - \frac{1}{6\sqrt{6}}\partial^2 U\right) + \frac{1}{3}\, c_{uotu}\, \partial\left(TU - \frac{1}{4}\partial^2 U\right) \right. \\ &\left. + \frac{1}{3}\partial P_{\frac{9}{2}} + Q_{\frac{11}{2}}\right], \end{aligned} \tag{3.20}$$

where the correct spin-4 current is the sum of previous spin-4 currents [15] with (3.9) and (3.15) as before

$$O_4(z) = \left(-\frac{1}{\sqrt{6}}P_{4'}^{uu} + \sqrt{6}P_{4'}^{ww}\right)(z). \tag{3.21}$$

This is identical to the relation (2.30). The structure constants in (3.20) are given by

$$\begin{aligned} c_{uou} &= \frac{6\sqrt{6}(-1+k)(10+k)(9+2k)^2}{(3+2k)(15+2k)(378+333k+37k^2)}, \\ c_{uogw} &= \frac{12(-1+k)(3+k)(6+k)(10+k)(9+2k)^2(1290+1197k+133k^2)}{5(3+2k)(15+2k)(74+45k+5k^2)(90+117k+13k^2)(378+333k+37k^2)}, \\ c_{uotu} &= \frac{8\sqrt{\frac{2}{3}}(-1+k)(3+k)(6+k)(10+k)(9+2k)^2(90+261k+29k^2)}{(3+2k)(15+2k)(74+45k+5k^2)(90+117k+13k^2)(378+333k+37k^2)}, \end{aligned} \tag{3.22}$$

where all of these have $(k-1)$ factors in their expressions. Of course, these constants (3.22) reduce to the ones appearing in (2.39) for infinite k limit.

[15] One has also the other spin-4 current $O_{4'}(z) = \frac{1}{8}\left(\frac{16}{7}\sqrt{\frac{2}{3}}P_{4'}^{uu} - \frac{4}{7}\sqrt{6}P_{4'}^{ww}\right)(z)$ which is equal to the relation (2.34) with (3.9) and (3.15).

Once again, the coefficients c_{uogw} and c_{uotu} appearing in the second-order pole are determined by evaluating the OPEs between $T(z)$(and similarly $G(z)$) and the whole second-order pole terms subtracted the first three terms where the coefficient c_{uou} is known from the higher order terms. The vanishing of higher order singular terms where the order is greater than 2 fixes the above unknown two coefficients. It turns out that they are given in (3.22). Then we are left with the primary field in this singular terms. The spin-$\frac{9}{2}$ primary current, which generalizes the previous expression (2.40), can be obtained

$$P_{\frac{9}{2}}(z) = \frac{n_1}{d_1} d^{abc} f^{bde} f^{cfg} \psi^a \psi^d \psi^e \psi^f \partial^2 \psi^g(z) + \text{other 4671 terms}, \tag{3.23}$$

where the intermediate k-dependent expressions are

$$\begin{aligned} n_1 &= 2i\sqrt{\frac{2}{5}}(-1+k)k(9+2k)(79110+149883k+86489k^2+19367k^3+1741k^4+50k^5), \quad (3.24) \\ d_1 &= 5(3+k)(15+2k)(74+45k+5k^2)(90+117k+13k^2)\sqrt{810+1053k+441k^2+72k^3+4k^4}. \end{aligned}$$

In this case, also only $\psi^a(z)$ dependent terms have $(k-1)$ factor in (3.23). Of course, the expressions (3.24) reduce to the numerical coefficient in (2.40).

What about the first-order singular terms? Since the second-order terms are determined, one finds their descendant fields with the known coefficient functions. Then by introducing the arbitrary three coefficient functions in (2.41), one can solve the equation such that the whole first-order terms subtracted by above four known descendant fields terms is equal to the quasi-primary field $Q_{\frac{11}{2}}(w)$. This provides all the information for the three unknown coefficient functions we introduced. Therefore, the general expression containing (2.41) is obtained

$$\begin{aligned} Q_{\frac{11}{2}}(z) &= -\frac{8\sqrt{\frac{2}{3}}(3+k)(6+k)(9+2k)^2(18+657k+73k^2)}{15(3+2k)(15+2k)(90+117k+13k^2)(378+333k+37k^2)} \\ &\times \left(T\partial U - \frac{5}{4}\partial T U - \frac{1}{7}\partial^3 U\right)(z) \\ &+ \frac{2(3+k)(6+k)(9+2k)^2(498+225k+25k^2)}{5(3+2k)(15+2k)(90+117k+13k^2)(378+333k+37k^2)} \\ &\times \left(G\partial W - 2\partial G W - \frac{1}{21}\sqrt{\frac{2}{3}}\partial^3 U\right)(z) \\ &+ \frac{3(3+k)(6+k)}{(3+2k)(15+2k)}\left(G O_{4''} - \frac{2}{9}\partial O_{\frac{9}{2}'}\right)(z). \end{aligned} \tag{3.25}$$

Compared to (2.41), one has k-dependent coefficient functions in front of three independent quasi-primary fields in (3.25). One can also compute the OPEs as in (2.43) and (2.44) and

they will show the explicit k-dependence in the right hand side of the OPEs. Compared to the previous OPE (3.16), in this example, we did not need to compute the OPEs between the $T(z)$(or $G(z)$) and some terms in the first-order pole. This is due to the fact that there are no other quasi-primary fields in the first-order terms. Here, the generalization of (2.42) is given by

$$\begin{aligned} O_{\frac{9}{2}'}(z) &= -\frac{8i\sqrt{\frac{2}{5}}(-1+k)k(1+k)(9+k)(9+2k)\sqrt{45+36k+4k^2}}{5(3+k)(3+2k)(15+2k)\sqrt{18+9k+k^2}(90+117k+13k^2)} \\ &\times\ d^{abc} f^{bde} f^{cfg} \psi^a \psi^d \psi^e \psi^f \partial^2 \psi^g(z) + \text{other 4430 terms}. \end{aligned} \tag{3.26}$$

Only $\psi^a(z)$ dependent terms have $(k-1)$ factor in (3.26).

Let us consider the OPE (2.45) when k is finite. It turns out that

$$\begin{aligned} & W(z)\, O_{\frac{7}{2}}(w) = \frac{1}{(z-w)^4}\, c_{wou}\, U(w) + \frac{1}{(z-w)^3}\, \frac{2}{5}\, c_{wou}\, \partial U(w) \\ & + \frac{1}{(z-w)^2} \left[\frac{1}{10}\, c_{wou}\, \partial^2 U + c_{wotu} \left(TU - \frac{1}{4}\partial^2 U \right) + c_{wogw} \left(GW - \frac{1}{6\sqrt{6}} \partial^2 U \right) + c_{wop}\, P_{\frac{9}{2}} \right. \\ & \left. + c_{woo'}\, O_{\frac{9}{2}'} \right](w) + \frac{1}{(z-w)} \left[\frac{2}{105}\, c_{wou}\, \partial^3 U + \frac{4}{9} c_{wotu} \partial \left(TU - \frac{1}{4}\partial^2 U \right) \right. \\ & \left. + \frac{4}{9}\, c_{wogw}\, \partial \left(GW - \frac{1}{6\sqrt{6}} \partial^2 U \right) + \frac{4}{9}\, c_{wop}\, \partial P_{\frac{9}{2}} + \frac{4}{9}\, c_{woo'}\, \partial O_{\frac{9}{2}'} + Q_{\frac{11}{2}'} \right](w) + \cdots. \end{aligned} \tag{3.27}$$

The three structure constants in (3.27) which depend on k explicitly are written in terms of

$$\begin{aligned} c_{wou} &= \frac{6(-1+k)(10+k)(9+2k)^2}{(3+2k)(15+2k)(378+333k+37k^2)}, \qquad c_{wop} = \frac{1}{\sqrt{6}}, \qquad c_{woo'} = \frac{1}{\sqrt{6}}, \\ c_{wotu} &= \frac{4(-1+k)(3+k)(6+k)(10+k)(9+2k)^2(846+927k+103k^2)}{(3(3+2k)(15+2k)(74+45k+5k^2)(90+117k+13k^2)(378+333k+37k^2)}, \\ c_{wogw} &= \frac{4\sqrt{6}(-1+k)(3+k)(6+k)(10+k)(9+2k)^2(90+261k+29k^2)}{5(3+2k)(15+2k)(74+45k+5k^2)(90+117k+13k^2)(378+333k+37k^2)}. \end{aligned} \tag{3.28}$$

In particular, they contain $(k-1)$ factor. One sees that these (3.28) become those in (2.45). Since the algebraic structure is known for infinite k, one takes four undetermined coefficients, $c_{wotu}, c_{wogw}, c_{wop}$ and $c_{woo'}$ in front of two quasi-primary fields and two primary fields (given by (3.23) and (3.26)) respectively. Note that the constant c_{wou} can be fixed from the higher order terms. Then one can write down the second order terms as in (3.27). On the other hand, one knows the explicit second-order pole from WZW currents. By equating these two, eventually one obtains the unknown four coefficient functions and they are given in (3.28).

Furthermore, the spin-$\frac{11}{2}$ quasi-primary field can be obtained. As we did in the OPE (3.20), due to the fact that there is no other quasi-primary field except this spin-$\frac{11}{2}$ current,

one can find the explicit form for this field as follows:

$$
\begin{aligned}
Q_{\frac{11}{2}'}(z) &= -\frac{2\sqrt{\frac{2}{3}}(3+k)(6+k)(9+2k)^2(18+657k+73k^2)}{15(3+2k)(15+2k)(90+117k+13k^2)(378+333k+37k^2)} \\
&\times \left(G\partial W - 2\partial G W - \frac{1}{21}\sqrt{\frac{2}{3}}\partial^3 U\right)(z) \\
&+ \frac{16(3+k)(6+k)(9+2k)^2(738+9k+k^2)}{135(3+2k)(15+2k)(90+117k+13k^2)(378+333k+37k^2)} \\
&\times \left(T\partial U - \frac{5}{4}\partial T U - \frac{1}{7}\partial^3 U\right)(z) \\
&+ \frac{\sqrt{\frac{3}{2}}(3+k)(6+k)}{(3+2k)(15+2k)}\left(G O_{4''} - \frac{2}{9}\partial O_{\frac{9}{2}'}\right)(z).
\end{aligned}
\tag{3.29}
$$

As before, one can also compute the OPEs as in (2.47) and (2.48). Even for the k-dependent coefficients, this field (3.29) is quasi-primary field because the three terms are quasi-primary fields.

For the OPE between the spin-3 current and the spin-4 current (3.21), one has the following k-dependent expression which appeared in (2.49)

$$
\begin{aligned}
& W(z)\,O_4(w) = \frac{1}{(z-w)^4}\,c_{wow}\,W(w) \\
& +\frac{1}{(z-w)^3}\left[\frac{1}{3}\,c_{wow}\,\partial W + c_{woo}\,O_{4''} + c_{wogu}\left(GU - \frac{\sqrt{6}}{3}\partial W\right)\right](w) \\
& +\frac{1}{(z-w)^2}\left[\frac{1}{14}c_{wow}\,\partial^2 W + \frac{3}{8}\,c_{woo}\,\partial O_{4''} + \frac{3}{8}\,c_{wogu}\,\partial\left(GU - \frac{\sqrt{6}}{3}\partial W\right) + c_{woo'}\,O_5 + Q_5\right](w) \\
& +\frac{1}{(z-w)}\left[\frac{1}{84}c_{wow}\,\partial^3 W + \frac{1}{12}\,c_{woo}\,\partial^2 O_{4''} + \frac{1}{12}\,c_{wogu}\,\partial^2\left(GU - \frac{\sqrt{6}}{3}\partial W\right)\right. \\
& \left. +\frac{2}{5}\,c_{woo'}\,\partial O_5 + \frac{2}{5}\,\partial Q_5 + Q_6\right](w) + \cdots .
\end{aligned}
\tag{3.30}
$$

The structure constants are given in terms of

$$
\begin{aligned}
c_{wow} &= \frac{48\sqrt{6}(-1+k)(10+k)(9+2k)^2}{5(3+2k)(15+2k)(378+333k+37k^2)}, \qquad c_{woo} = \sqrt{\frac{2}{3}}, \qquad c_{woo'} = 5\sqrt{\frac{2}{3}}, \\
c_{wogu} &= \frac{12(-1+k)(3+k)(6+k)(10+k)(9+2k)^2}{(3+2k)(15+2k)(90+117k+13k^2)(378+333k+37k^2)}.
\end{aligned}
\tag{3.31}
$$

As before, the coefficient c_{wow} can be fixed easily from the fourth-order pole. Since the algebraic structure is determined from the infinite k result, one puts two unknown coefficients c_{woo} and c_{wogu} in the two primary fields respectively. One of the primary fields was given

in (3.18). Then these two coefficients can be fixed without any difficulty. Now we move on the next order singular terms. The easiest way to obtain the quasi-primary fields in (3.30) is that for example, once we find the second-order pole in (3.30), then one puts the arbitrary coefficient function $c_{woo'}$ and two additional coefficients in the quasi-primary field in (2.50). Then one can fix all the coefficients as in (3.31) and the spin-5 quasi-primary field with k-dependent coefficients is given by

$$
\begin{aligned}
Q_5(z) &= \frac{2\sqrt{\frac{2}{3}}(9+2k)^2(-168156+101412k+104013k^2+20610k^3+1145k^4)}{45(3+2k)(15+2k)(74+45k+5k^2)(378+333k+37k^2)} \\
&\times \left(TW - \frac{3}{14}\partial^2 W\right)(z) \\
&- \frac{(3+k)(6+k)(9+2k)^2(-5166+855k+95k^2)}{90(3+2k)(15+2k)(74+45k+5k^2)(378+333k+37k^2)} \\
&\times \left(G\partial U - \frac{5}{3}\partial G U - \frac{\sqrt{6}}{7}\partial^2 W\right)(z).
\end{aligned}
\tag{3.32}
$$

Similarly, one can analyze for the spin-6 quasi-primary field as done in (3.32) and one puts the four unknown coefficients in (2.52). One can construct the OPEs as in (2.51). By equating the first-order pole to the above expressions with four unknown coefficients, one obtains all of them. It turns out that one has the following spin-6 quasi-primary field

$$
\begin{aligned}
Q_6(z) &= \frac{32(-1+k)(3+k)(6+k)(10+k)(9+2k)^2}{25(3+2k)(15+2k)(90+117k+13k^2)(378+333k+37k^2)} \\
&\times \left(G\partial^2 U - 4\partial G\partial U + \frac{5}{2}\partial^2 G U - \frac{1}{2\sqrt{6}}\partial^3 W\right)(z) \\
&- \frac{192\sqrt{6}(-1+k)(3+k)(6+k)(10+k)(9+2k)^2}{25(3+2k)(15+2k)(90+117k+13k^2)(378+333k+37k^2)} \\
&\times \left(T\partial W - \frac{3}{2}\partial T W - \frac{1}{8}\partial^3 W\right)(z) \\
&+ \frac{\sqrt{6}(3+k)(6+k)}{(3+2k)(15+2k)}\left(TO_{4''} - \frac{1}{6}\partial^2 O_{4''}\right)(z) \\
&- \frac{\sqrt{\frac{3}{2}}(3+k)(6+k)}{(3+2k)(15+2k)}\left(GO_{\frac{9}{2}'} - \frac{1}{9}\partial^2 O_{4''}\right)(z).
\end{aligned}
\tag{3.33}
$$

The OPEs between the spin-2 current (or the spin-$\frac{3}{2}$ current) and the current (3.33) can be computed as in (2.53).

As in infinite k case (2.54), one has the following relation with (3.23) and (3.26)

$$
O_{\frac{9}{2}''}(z) = O_{\frac{9}{2}'}(z) + \frac{8}{5}P_{\frac{9}{2}}(z).
\tag{3.34}
$$

Therefore, as for infinite k case, one finds the two primary currents (3.18) and (3.19). From the other two primary fields (3.23) and (3.26), one constructs the primary current (3.34). In other words, the four independent currents in (2.20) are found during the computations of the OPEs (3.16), (3.20), (3.27) and (3.30). One sees four quasi-primary fields that can be written in terms of known higher spin currents as well as the stress energy tensor (and its superpartner).

• The construction of $\hat{O}_{\frac{11}{2}}(Z)$ and $\hat{O}_6(Z)$

One continues to compute the OPE between the spin-$\frac{5}{2}$ current (3.6) and the spin-4 current (3.18), corresponding to the infinite k result (2.55),

$$
\begin{aligned}
& U(z)\, O_{4''}(w) = \frac{1}{(z-w)^3}\, c_{uoo}\, O_{\frac{7}{2}}(w) + \frac{1}{(z-w)^2} \left[\frac{2}{7}\, c_{uoo}\, \partial O_{\frac{7}{2}} + c_{uoo'}\, O_{\frac{9}{2}} \right](w) \\
& + \frac{1}{(z-w)} \left[\frac{3}{56}\, c_{uoo}\, \partial^2 O_{\frac{7}{2}} - \frac{1}{3}\, c_{uoo'}\, \partial O_{\frac{9}{2}} + c_{uoto} \left(T O_{\frac{7}{2}} - \frac{3}{16} \partial^2 O_{\frac{7}{2}} \right) \right. \\
& + c_{uogp} \left(G P^{uu}_{4'} - \frac{4\sqrt{6}}{9} \partial O_{\frac{9}{2}} - \frac{\sqrt{6}}{56} \partial^2 O_{\frac{7}{2}} \right) - c_{uogp'} \left(G P^{ww}_{4'} - \frac{2}{9}\sqrt{\frac{2}{3}} \partial O_{\frac{9}{2}} - \frac{1}{14}\sqrt{\frac{2}{3}} \partial^2 O_{\frac{7}{2}} \right) \\
& \left. + O_{\frac{11}{2}} \right](w) + \cdots . \qquad (3.35)
\end{aligned}
$$

The structure constants in (3.35) can be obtained

$$
\begin{aligned}
c_{uoo} & = \frac{24(1+k)(8+k)(378+333k+37k^2)}{5(3+k)(6+k)(90+117k+13k^2)}, \\
c_{uogw} & = -\frac{2(9+2k)^2(-42+99k+11k^2)}{5(3+k)(6+k)(90+117k+13k^2)}, \\
c_{uoto} & = \frac{288(1+k)(8+k)(846+585k+65k^2)}{5(90+117k+13k^2)(954+549k+61k^2)}, \\
c_{uogp} & = \frac{n_1}{d}, \qquad c_{uogp'} = \frac{n_2}{d}, \qquad (3.36)
\end{aligned}
$$

where the numerators and denominator in the last two coefficients are functions of k

$$
\begin{aligned}
n_1 & \equiv 2\sqrt{6}(2135484 + 3378672k + 2386233k^2 + 869670k^3 + 165765k^4 + 15660k^5 + 580k^6), \\
n_2 & \equiv -3\sqrt{\frac{3}{2}}(13838364 + 31369356k + 26544483k^2 + 10463418k^3 + 2064411k^4 \\
& + 197748k^5 + 7324k^6), \\
d & \equiv 5(3+2k)(15+2k)(90+117k+13k^2)(954+549k+61k^2). \qquad (3.37)
\end{aligned}
$$

The OPEs between the spin-2 and spin-$\frac{3}{2}$ currents, $T(z)$ and $G(z)$, with the quasi-primary fields appearing in (3.35) can be found in the Appendices D and E, as we described before. It is straightforward to determine the structure of the third- and second-order poles, as we did

before. In the first-order term, the first two descendant terms are known and one introduces three unknown coefficients c_{uoto}, c_{uogp} and $c_{uogp'}$. Then how does one obtain these k-dependent coefficients? Since one expects that the remaining term is characterized by the primary field of spin-$\frac{11}{2}$, one requires that there should be only the second-order and first-order terms when one computes the OPE between the currents $T(z)$ or $G(z)$ and the first-order pole subtracted the above five terms including two derivative terms. This condition fixes the above unknown three coefficients and they are given in (3.36) and (3.37) finally.

Then we are left with the spin-$\frac{11}{2}$ primary current which is given by

$$
\begin{aligned}
O_{\frac{11}{2}}(z) &= \frac{n_1}{d_1} f^{abc} \psi^a \psi^b \partial^4 \psi^c(z) + \text{other 22096 terms}, \\
n_1 &\equiv 3\sqrt{2}(-1+k)k(1+k)(9+k)(9+2k) \\
&\times (-23328 - 31338k - 12843k^2 - 1370k^3 + 53k^4 + 10k^5), \\
d_1 &\equiv 10(3+k)(6+k)(3+2k)(15+2k)^2 \sqrt{18+9k+k^2}(90+117k+13k^2) \\
&\times (954+549k+61k^2),
\end{aligned}
\tag{3.38}
$$

which generalizes the previous expression (2.56) for the infinite k.

Let us describe the next OPE. The OPE between the spin-$\frac{5}{2}$ current (3.6) and the spin-$\frac{9}{2}$ current (3.26), corresponding to the previous result (2.57), for the finite k is summarized by

$$
\begin{aligned}
& U(z)\, O_{\frac{9}{2}'}(w) = \frac{1}{(z-w)^3} \left[c_{uop}\, P_{4'}^{uu} - c_{uop'} P_{4'}^{ww} \right](w) \\
& + \frac{1}{(z-w)^2} \left[\frac{1}{4} c_{uop}\, \partial P_{4'}^{uu} - \frac{1}{6} c_{uop'}\, \partial P_{4'}^{ww} + c_{uogo} \left(G O_{\frac{7}{2}} + \frac{1}{4\sqrt{6}} \partial P_{4'}^{uu} - \frac{\sqrt{6}}{4} \partial P_{4'}^{ww} \right) \right](w) \\
& + \frac{1}{(z-w)} \left[\frac{1}{24} c_{uop}\, \partial^2 P_{4'}^{uu} - \frac{1}{24} c_{uop'}\, \partial^2 P_{4'}^{ww} + \frac{3}{10} c_{uogo}\, \partial \left(G O_{\frac{7}{2}} + \frac{1}{4\sqrt{6}} \partial P_{4'}^{uu} - \frac{\sqrt{6}}{4} \partial P_{4'}^{ww} \right) \right. \\
& + c_{uotp} \left(T P_{4'}^{uu} - \frac{1}{6} \partial^2 P_{4'}^{uu} \right) - c_{uotp'} \left(T P_{4'}^{ww} - \frac{1}{6} \partial^2 P_{4'}^{ww} \right) \\
& - c_{uogo'} \left(G \partial O_{\frac{7}{2}} - \frac{7}{3} \partial G O_{\frac{7}{2}} + \frac{1}{9\sqrt{6}} \partial^2 P_{4'}^{uu} - \frac{1}{3} \sqrt{\frac{2}{3}} \partial^2 P_{4'}^{ww} \right) \\
& \left. - c_{uogo''} \left(G O_{\frac{9}{2}} - \frac{2}{63} \sqrt{\frac{2}{3}} \partial^2 P_{4'}^{uu} + \frac{1}{21\sqrt{6}} \partial^2 P_{4'}^{ww} \right) + P_6 \right](w) + \cdots .
\end{aligned}
\tag{3.39}
$$

The structure constants in (3.39) are given by

$$
\begin{aligned}
c_{uop} &= \frac{4\sqrt{\frac{2}{3}}(10368 + 5562k + 2967k^2 + 522k^3 + 29k^4)}{5(3+k)(6+k)(90+117k+13k^2)}, \\
c_{uop'} &= \frac{8\sqrt{6}(6966 + 12069k + 7254k^2 + 1314k^3 + 73k^4)}{5(3+k)(6+k)(90+117k+13k^2)},
\end{aligned}
$$

$$
\begin{aligned}
c_{uogo} &= \frac{792(1+k)(8+k)}{5(90+117k+13k^2)}, \qquad c_{uotp} = \frac{n_1}{d}, \qquad c_{uotp'} = \frac{n_2}{d}, \\
c_{uogo'} &= -\frac{12(1+k)(8+k)(378+333k+37k^2)(1818+873k+97k^2)}{25(-6+9k+k^2)(90+117k+13k^2)(954+549k+61k^2)}, \\
c_{uogo''} &= -\frac{21(9+2k)^2(66+45k+5k^2)(-42+99k+11k^2)}{5(3+2k)(15+2k)(90+117k+13k^2)(366+207k+23k^2)}, \qquad (3.40)
\end{aligned}
$$

where the numerators and denominator in (3.40) of fourth and fifth constants are given by

$$
\begin{aligned}
n_1 &\equiv 28\sqrt{\frac{2}{3}}(830045232 + 8372710800k + 18353902752k^2 + 19040042412k^3 + 11562105645k^4 \\
&+ 4408598988k^5 + 1070601270k^6 + 164251116k^7 + 15373601k^8 + 800820k^9 + 17796k^{10}), \\
n_2 &\equiv -7\sqrt{6}(9794995632 + 61010651520k + 137265126552k^2 + 158181370992k^3 \\
&+ 105657645375k^4 + 42990128568k^5 + 10861840470k^6 + 1705114836k^7 \\
&+ 161571871k^8 + 8459820k^9 + 187996k^{10}), \\
d &\equiv 5(3+2k)(15+2k)(-6+9k+k^2)(90+117k+13k^2)(366+207k+23k^2) \\
&\times (954+549k+61k^2). \qquad (3.41)
\end{aligned}
$$

The third-order pole in the right hand side can be easily determined. The coefficient for the quasi-primary field in the second-order pole can be fixed without any difficulty. How does one make sure that there exist four quasi-primary fields and a single primary field in the last first-order term? Since the derivative terms are completely fixed, one can introduce four unknown coefficients. We follow the procedure what we have done for infinite k case. Then these four coefficients are determined. Finally, after subtracting these four quasi-primary fields correctly, we are left with the following new spin-6 primary field. The spin-6 current is given by

$$
\begin{aligned}
P_6(z) &= \frac{n_1}{d}\, \psi^a \partial^5 \psi^a(z) + \text{other lower order derivative terms}, \\
n_1 &\equiv (-1+k)k(1+k)(9+k)(9+2k)(-1365527808 - 3366282888k - 3098773908k^2 \\
&- 1210990014k^3 - 58275207k^4 + 110774898k^5 + 38348106k^6 + 5639106k^7 + 398009k^8 \\
&+ 11050k^9), \\
d &\equiv 75\sqrt{6}(3+k)(6+k)^2(3+2k)(15+2k)^2(-6+9k+k^2)(90+117k+13k^2) \\
&\times (366+207k+23k^2)(954+549k+61k^2). \qquad (3.42)
\end{aligned}
$$

As in previous case, the factor $(k-1)$ is contained in this $\psi^a(z)$-dependent spin-6 current.

Similarly, one has the following OPE between the spin-3 current (3.5) and the spin-4

current (3.18), corresponding to the previous result (2.59),

$$
\begin{aligned}
& W(z)\, O_{4''}(w) = \frac{1}{(z-w)^3}\, \left[-c_{wop}\, P^{uu}_{4'} + c_{wop'}\, P^{ww}_{4'}\right](w) \\
& + \frac{1}{(z-w)^2} \left[-\frac{3}{8}\, c_{wop}\, \partial P^{uu}_{4'} + \frac{3}{8}\, c_{wop'}\, \partial P^{ww}_{4'} + c_{wogo} \left(GO_{\frac{7}{2}} + \frac{1}{4\sqrt{6}} \partial P^{uu}_{4'} - \frac{\sqrt{6}}{4} \partial P^{ww}_{4'} \right)\right](w) \\
& + \frac{1}{(z-w)} \left[-\frac{1}{12}\, c_{wop}\, \partial^2 P^{uu}_{4'} + \frac{1}{12}\, c_{wop'}\, \partial^2 P^{ww}_{4'} + \frac{2}{5}\, c_{wogo}\, \partial \left(GO_{\frac{7}{2}} + \frac{1}{4\sqrt{6}} \partial P^{uu}_{4'} - \frac{\sqrt{6}}{4} \partial P^{ww}_{4'} \right) \right. \\
& - c_{wotp} \left(T P^{uu}_{4'} - \frac{1}{6} \partial^2 P^{uu}_{4'} \right) - c_{wotp'} \left(T P^{ww}_{4'} - \frac{1}{6} \partial^2 P^{ww}_{4'} \right) \\
& - c_{wogo'} \left(G \partial O_{\frac{7}{2}} - \frac{7}{3} \partial G O_{\frac{7}{2}} + \frac{1}{9\sqrt{6}} \partial^2 P^{uu}_{4'} - \frac{1}{3}\sqrt{\frac{2}{3}} \partial^2 P^{ww}_{4'} \right) \\
& \left. - c_{wogo''} \left(G O_{\frac{9}{2}} - \frac{2}{63}\sqrt{\frac{2}{3}} \partial^2 P^{uu}_{4'} + \frac{1}{21\sqrt{6}} \partial^2 P^{ww}_{4'} \right) + P_{6'} \right](w) + \cdots . \qquad (3.43)
\end{aligned}
$$

The unknown coefficients in (3.43) are fixed by

$$
\begin{aligned}
c_{wop} & = -\frac{8(-36+369k+41k^2)}{15(90+117k+13k^2)}, \\
c_{wop'} & = \frac{8(2106+6129k+3759k^2+684k^3+38k^4)}{5(3+k)(6+k)(90+117k+13k^2)}, \\
c_{wogo} & = \frac{132\sqrt{6}(1+k)(8+k)}{5(90+117k+13k^2)}, \qquad c_{wotp} = \frac{n_1}{d}, \qquad c_{wotp'} = \frac{n_2}{d}, \\
c_{wogo'} & = -\frac{64\sqrt{6}(1+k)(8+k)(369+234k+26k^2)(378+333k+37k^2)}{175(-6+9k+k^2)(90+117k+13k^2)(954+549k+61k^2)}, \\
c_{wogo''} & = -\frac{6\sqrt{6}(1+k)(8+k)(9+2k)^2(-42+99k+11k^2)}{5(3+2k)(15+2k)(90+117k+13k^2)(366+207k+23k^2)}, \\
n_1 & \equiv -\frac{32}{3}(3+k)(6+k)(-109854468-570043980k-583026525k^2-175058712k^3 \\
& + 15297117k^4+16269390k^5+2997578k^6+228096k^7+6336k^8), \\
n_2 & \equiv -4(42504047568+157399614720k+224326680552k^2+165246862320k^3 \\
& + 70324852617k^4+18329683752k^5+3051110298k^6+334279692k^7+24334537k^8 \\
& + 1114740k^9+24772k^{10}), \qquad (3.44)
\end{aligned}
$$

where d in (3.44) is the same as the one in (3.41). In order to arrive at the final result (3.43), the OPEs between the spin-2 and spin-$\frac{3}{2}$ currents with the quasi-primary fields appearing in (3.43) should be obtained. They can be found in the Appendices D and E. The structure constants on the two spin-4 fields and its descendant fields are completely fixed. The nontrivial second-order pole can be determined similarly. Based on the results for the second-order

pole, one obtains the correct three derivative terms in the first-order singular terms and one can easily determine the coefficients appearing in the quasi-primary fields as before. After extracting these four quasi-primary fields from the first-order pole, one arrives at the following new spin-6 primary field

$$\begin{aligned}
P_{6'}(z) &= \frac{n_1}{d}\,\psi^a \partial^5 \psi^a(z) + \text{other lower order derivative terms}, \\
n_1 &\equiv 2(-1+k)k(1+k)(9+k)(9+2k)(-114528816 - 144741492k - 9500652k^2 \\
&+ 61972749k^3 + 35947494k^4 + 8541978k^5 + 996804k^6 + 56069k^7 + 1210k^8), \\
d &\equiv 225(3+k)(6+k)(3+2k)(15+2k)^2(-6+9k+k^2)(90+117k+13k^2) \\
&\times (366+207k+23k^2)(954+549k+61k^2).
\end{aligned} \tag{3.45}$$

Then one can construct the following spin-6 current which is a superpartner of the spin-$\frac{11}{2}$ current (3.38), together with (3.42) and (3.45),

$$O_6(z) = -P_6(z) + \sqrt{6} P_{6'}(z), \tag{3.46}$$

which is the same as (2.61).

Now it is ready to compute the last final OPE which is more involved.

$$\begin{aligned}
& W(z)\, O_{\frac{9}{2}'}(w) = \frac{1}{(z-w)^4}\, c_{woo}\, O_{\frac{7}{2}} + \frac{1}{(z-w)^3}\left[\frac{2}{7}\, c_{woo}\, \partial O_{\frac{7}{2}} - c_{woo'}\, O_{\frac{9}{2}}\right](w) \\
& + \frac{1}{(z-w)^2}\left[\frac{3}{56}\, c_{woo}\, \partial^2 O_{\frac{7}{2}} - \frac{1}{3}\, c_{woo'}\, \partial O_{\frac{9}{2}} + c_{woo''}\, O_{\frac{11}{2}}\right. \\
& + c_{woto}\left(T O_{\frac{7}{2}} - \frac{3}{16}\partial^2 O_{\frac{7}{2}}\right) + c_{wogp}\left(G P^{uu}_{4'} - \frac{4\sqrt{6}}{9}\partial O_{\frac{9}{2}} - \frac{\sqrt{6}}{56}\partial^2 O_{\frac{7}{2}}\right) \\
& \left. - c_{wogp'}\left(G P^{ww}_{4'} - \frac{2}{9}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}} - \frac{1}{14}\sqrt{\frac{2}{3}}\partial^2 O_{\frac{7}{2}}\right)\right](w) \\
& + \frac{1}{(z-w)}\left[\frac{1}{126}\, c_{woo}\, \partial^3 O_{\frac{7}{2}} - \frac{1}{15} c_{woo'}\, \partial^2 O_{\frac{9}{2}} + \frac{4}{11}\, c_{woo''}\, \partial O_{\frac{11}{2}}\right. \\
& + \frac{4}{11}\, c_{woto}\, \partial\left(T O_{\frac{7}{2}} - \frac{3}{16}\partial^2 O_{\frac{7}{2}}\right) + \frac{4}{11}\, c_{wogp}\, \partial\left(G P^{uu}_{4'} - \frac{4\sqrt{6}}{9}\partial O_{\frac{9}{2}} - \frac{\sqrt{6}}{56}\partial^2 O_{\frac{7}{2}}\right) \\
& - \frac{4}{11}\, c_{wogp'}\, \partial\left(G P^{ww}_{4'} - \frac{2}{9}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}} - \frac{1}{14}\sqrt{\frac{2}{3}}\partial^2 O_{\frac{7}{2}}\right) \\
& - c_{woto'}\left(T O_{\frac{9}{2}} - \frac{3}{20}\partial^2 O_{\frac{9}{2}}\right) - c_{woto''}\left(T \partial O_{\frac{7}{2}} - \frac{7}{4}\partial T O_{\frac{7}{2}} - \frac{1}{9}\partial^3 O_{\frac{7}{2}}\right) \\
& - c_{wogp''}\left(G \partial P^{uu}_{4'} - \frac{8}{3}\partial G P^{uu}_{4'} - \frac{\sqrt{6}}{5}\partial^2 O_{\frac{9}{2}} - \frac{2}{63}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right)
\end{aligned}$$

$$+c_{wogp'''}\left(G\partial P^{ww}_{4'}-\frac{8}{3}\partial G P^{ww}_{4'}-\frac{1}{5\sqrt{6}}\partial^2 O_{\frac{9}{2}}-\frac{8}{189}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right)+\frac{2}{11}\sqrt{\frac{2}{3}}O_{\frac{13}{2}}\Bigg](w)$$
$$+\cdots. \tag{3.47}$$

The structure constants in (3.47) are given by

$$\begin{aligned}
c_{woo} &= \frac{12\sqrt{6}(1+k)(8+k)(378+333k+37k^2)}{5(3+k)(6+k)(90+117k+13k^2)},\\
c_{woo'} &= -\frac{2\sqrt{\frac{2}{3}}(9+2k)^2(-42+99k+11k^2)}{(3+k)(6+k)(90+117k+13k^2)}, \qquad c_{woo''}=\frac{1}{\sqrt{6}},\\
c_{woto} &= \frac{216\sqrt{6}(1+k)(8+k)(1354+801k+89k^2)}{5(90+117k+13k^2)(954+549k+61k^2)}, \qquad c_{wogp}=\frac{n_1}{d}, \qquad c_{wogp'}=\frac{n_2}{d},\\
c_{woto'} &= -\frac{56\sqrt{\frac{2}{3}}(9+2k)^2(-42+99k+11k^2)}{5(90+117k+13k^2)(366+207k+23k^2)},\\
c_{woto''} &= -\frac{64\sqrt{\frac{2}{3}}(1+k)(8+k)(378+333k+37k^2)}{55(-6+9k+k^2)(90+117k+13k^2)},\\
c_{wogp''} &= -\frac{56(152280+426492k+372846k^2+151785k^3+30505k^4+2943k^5+109k^6)}{55(-6+9k+k^2)(90+117k+13k^2)(366+207k+23k^2)},\\
c_{wogp'''} &= \frac{84(629532+1636308k+1520823k^2+647478k^3+133171k^4+12960k^5+480k^6)}{55(-6+9k+k^2)(90+117k+13k^2)(366+207k+23k^2)},\\
n_1 &\equiv 4(24802524+60056964k+54194319k^2+22722930k^3+4640895k^4\\
&+\ 450468k^5+16684k^6),\\
n_2 &\equiv -3(195174684+484795692k+441009171k^2+185289498k^3+37865451k^4\\
&\quad +3676212k^5+136156k^6),\\
d &\equiv 10(3+2k)(15+2k)(90+117k+13k^2)(954+549k+61k^2).
\end{aligned} \tag{3.48}$$

How does one obtain the complete set of structure constants? In principle, one can follow what one has done for infinite k case. However, the first-order pole is rather complicated. First of all, one should compute the first-order singular terms completely. It took too much time. After that, the derivative terms appearing in the first-order terms are known from the higher-order singular terms. Then we are left with four quasi-primary fields and one additional primary field of spin-$\frac{13}{2}$ which is the last element of the higher spin current in the list (2.20).

How does one determine the unknown k-dependent coefficient functions and explicit form the spin-$\frac{13}{2}$? One way to determine these is to write down the spin-$\frac{13}{2}$ current using its superpartner $O_{6'}(z)$. Since one has $P_6(z)$ and $P_{6'}(z)$ from (3.39) and (3.43) explicitly, the

current $O_{6'}(z)$ is determined by, similar to (3.46),

$$O_{6'}(z) = P_6(z) + \frac{7}{2}\sqrt{\frac{3}{2}} P_{6'}(z), \tag{3.49}$$

which is the same as (2.64). Then by equating the first-order pole from WZW currents to the sum of derivative terms, quasi-primary fields with four unknown coefficients and spin-$\frac{13}{2}$ current, the four unknown coefficients are fixed as in (3.48). The last spin-$\frac{13}{2}$ current which is a superpartner of (3.49) is given by

$$\begin{aligned}
O_{\frac{13}{2}}(z) &= \frac{n}{d} f^{abc}\psi^a\psi^b\partial^5\psi^c(z) + \text{other lower order derivative terms}, \\
n &\equiv 16\sqrt{2}(-1+k)k(1+k)(9+k)(9+2k)^2(-1998-123k+573k^2+103k^3+5k^4), \\
d &\equiv 75(3+k)(6+k)(15+2k)(-6+9k+k^2)\sqrt{18+9k+k^2}(90+117k+13k^2) \\
&\times (366+207k+23k^2).
\end{aligned} \tag{3.50}$$

The OPEs between the spin-2 and spin-$\frac{3}{2}$ currents with the seven quasi-primary fields appearing in (3.47) can be found in the Appendices D and E.

Therefore, the higher spin currents are given by (3.6), (3.5), (3.11), (3.12), (3.18), (3.19), (3.21), footnote 15, (3.26), (3.34), (3.38), (3.46), (3.49), and (3.50) and some of the OPEs between them are computed. In these computations, the infinite k case done in previous section is very crucial because the algebraic structure in the OPEs is common to each other. Although one obtains all the singular terms in the OPEs from the defining OPE between the current $\psi^a(z)$ and the current $K^a(z)$, it was nontrivial to write down those singular terms in terms of the quasi-primary fields and higher spin currents. Note that the quasi-primary fields can be expressed as those higher spin currents and the stress energy tensor with its superpartner. We would expect that the right hand side of the remaining OPEs we did not consider should contain only the known primary currents (and their composite fields that can be either quasi-primary fields or primary fields) in the list (2.20).

4 Conclusions and outlook

We have found that the higher spin currents in the list of (2.20) for the $c \leq 4$ model in terms of WZW currents $\psi^a(z)$ and $K^a(z)$. They satisfy the following superfusion rules

$$\begin{aligned}
\left[\hat{W}\right]\left[\hat{W}\right] &= \left[\hat{I}\right] + \left[\hat{O}_{\frac{7}{2}}\right] + \left[\hat{O}_4\right], \\
\left[\hat{W}\right]\left[\hat{O}_{\frac{7}{2}}\right] &= \left[\hat{W}\right] + \left[\hat{O}_{4'}\right] + \left[\hat{O}_{\frac{9}{2}}\right], \\
\left[\hat{W}\right]\left[\hat{O}_{4'}\right] &= \left[\hat{O}_{\frac{7}{2}}\right] + \left[\hat{O}_4\right] + \left[\hat{O}_{\frac{11}{2}}\right] + \left[\hat{O}_6\right].
\end{aligned} \tag{4.1}$$

All the coefficients in the OPEs (4.1) are fixed. In the third superfusion rule, some of the algebraic structure of the first one occur. We did not compute the other remaining OPEs. According to the observation of [9], they will, in general, take the form $\left[\hat{I}\right] + \left[\hat{W}\right] + \left[\hat{O}_{\frac{7}{2}}\right] + \left[\hat{O}_4\right] + \left[\hat{O}_{4'}\right] + \left[\hat{O}_{\frac{9}{2}}\right] + \left[\hat{O}_{\frac{11}{2}}\right] + \left[\hat{O}_6\right]$ in the right hand side. It is nontrivial task to find out the correct quasi-primary fields for given spins in terms of 16 currents. For the most complicated OPE between the spin-$\frac{13}{2}$ current and itself, the singular terms have 13-th order pole, $\cdots$, second-order pole, and first-order pole. Then the highest quasi-primary field of spin-12 can appear in the first-order singular term and should be written in terms of the known higher spin currents. Maybe the formula (B.4) will be helpful to find out the structure of this quasi-primary field.

One expects that for the general N, one can think of the following coset models with given central charge

$$\frac{\widehat{SU}(N)_k \oplus \widehat{SU}(N)_N}{\widehat{SU}(N)_{k+N}}, \qquad c = \frac{(N^2-1)}{2}\left[1 - \frac{2N^2}{(k+N)(k+2N)}\right] < \frac{(N^2-1)}{2}. \tag{4.2}$$

This will be the supersymmetric extension of W_N algebra. As pointed out in [8], the lowest model in the series of coset models ($k = 1$ or $c = \frac{(-1+N)(1+3N)}{2(1+2N)}$) has "minimal" super W_N algebra where there exist the supercurrents of spins $\frac{3}{2}, \frac{5}{2}, \cdots, (N - \frac{1}{2})$. For the general $k > 1$, one expects that the extra additional currents should appear. As in the present paper, it is better to look at the $k \rightarrow \infty$ model (or $c = \frac{(N^2-1)}{2}$ fermion model in the adjoint representation of $SU(N)$) first because it contains all the algebraic structures and is simpler than the more general $c < \frac{(N^2-1)}{2}$ model. Then all the OPEs appeared in sections 2 and 3 should be generalized to the OPEs with N-dependence explicitly. In the context of minimal model holography [1, 2], one should compute the correct normalizations for the higher spin currents with spins greater than 3. In the original paper [9], the character technique was used to generate the complete currents in $c = 4$ eight fermion model. It would be interesting to generalize this for the arbitrary N. As a first step, it is also useful to consider the $N = 4$ case.

It would be interesting to study the most general coset models by $\frac{\widehat{SU}(N)_k \oplus \widehat{SU}(N)_l}{\widehat{SU}(N)_{k+l}}$ with central charge. For the particular case ($l = N$), this model reduces to the above model (4.2). For the higher spin $3, 4$ currents, the explicit construction was computed as mentioned in the introduction. It is an open problem to obtain other higher spin currents explicitly. For example, the spin-5 Casimir operator.

In [43], the coset model was based on the group $SO(N)$ with given central charge. It is straightforward to ask whether the present result can be applied to the different minimal model or not.

Acknowledgments

CA would like to thank his previous collaborators K. Schoutens and A. Sevrin for the work of [9]. This work was supported by the Mid-career Researcher Program through the National Research Foundation of Korea (NRF) grant funded by the Korean government (MEST) (No. 2012-045385). CA acknowledges warm hospitality from the School of Liberal Arts (and Institute of Convergence Fundamental Studies), Seoul National University of Science and Technology.

Appendix A The coefficients appearing in the descendant fields of quasi-primary or primary fields (1.3)

In the introduction, we presented the OPE between the two quasi-primary fields (including the primary fields) where the coefficient functions which depend on the spins h_i, h_j, h_k and the number of derivatives n are given by

$$A_{i,j,k,n} \equiv \frac{1}{n!}\frac{\Gamma(h_i-h_j+h_k+n)}{\Gamma(h_i-h_j+h_k)}\frac{\Gamma(2h_k)}{\Gamma(2h_k+n)} = \frac{1}{n!}\prod_{x=0}^{n-1}\frac{(h_i-h_j+h_k+x)}{(2h_k+x)}. \tag{A.1}$$

Here, we write them in terms of Pochhammer symbol because sometimes the denominator in the original expression can have zero values. In order to avoid this feature, one should write down the ratio of two Gamma functions in terms of Pochhammer symbol as in (A.1).

Using the definition of the coefficients (A.1), one can check those vanishing and nonvanishing coefficients appearing in the sections 2, and 3, as follows:

$$\begin{aligned}
A_{2,2,2,1} &= \frac{1}{2}, \quad A_{2,\frac{3}{2},\frac{3}{2},1} = \frac{2}{3}, \quad A_{\frac{3}{2},3,\frac{5}{2},1} = \frac{1}{5}, \quad A_{\frac{3}{2},4,\frac{7}{2},1} = \frac{1}{7},\\
A_{\frac{3}{2},\frac{9}{2},4,1} &= \frac{1}{8}, \quad A_{3,3,2,1} = \frac{1}{2}, \quad A_{3,3,2,2} = \frac{3}{20}, \quad A_{3,3,2,3} = \frac{1}{30},\\
A_{3,3,4,1} &= \frac{1}{2}, \quad A_{3,\frac{5}{2},\frac{3}{2},1} = \frac{2}{3}, \quad A_{3,\frac{5}{2},\frac{3}{2},2} = \frac{1}{4}, \quad A_{3,\frac{5}{2},\frac{3}{2},3} = \frac{1}{15},\\
A_{3,\frac{5}{2},\frac{7}{2},1} &= \frac{4}{7}, \quad A_{\frac{5}{2},\frac{5}{2},2,1} = \frac{1}{2}, \quad A_{\frac{5}{2},\frac{5}{2},2,2} = \frac{3}{20}, \quad A_{\frac{5}{2},\frac{7}{2},3,1} = \frac{1}{3},\\
A_{\frac{5}{2},\frac{7}{2},3,2} &= \frac{1}{14}, \quad A_{\frac{5}{2},\frac{7}{2},4,1} = \frac{3}{8}, \quad A_{\frac{5}{2},4,\frac{5}{2},1} = \frac{1}{5}, \quad A_{\frac{5}{2},4,\frac{5}{2},2} = \frac{1}{30},\\
A_{\frac{5}{2},4,\frac{5}{2},3} &= \frac{1}{210}, \quad A_{\frac{5}{2},4,\frac{9}{2},1} = \frac{1}{3}, \quad A_{2,\frac{11}{2},\frac{5}{2},1} = -\frac{1}{5}, \quad A_{\frac{3}{2},\frac{11}{2},3,1} = -\frac{1}{6},\\
A_{3,\frac{7}{2},\frac{5}{2},1} &= \frac{2}{5}, \quad A_{3,\frac{7}{2},\frac{5}{2},2} = \frac{1}{10}, \quad A_{3,\frac{7}{2},\frac{5}{2},3} = \frac{2}{105}, \quad A_{3,\frac{7}{2},\frac{9}{2},1} = \frac{4}{9},\\
A_{3,4,3,1} &= \frac{1}{3}, \quad A_{3,4,3,2} = \frac{1}{14}, \quad A_{3,4,3,3} = \frac{1}{84}, \quad A_{3,4,4,1} = \frac{3}{8},\\
A_{3,4,4,2} &= \frac{1}{12}, \quad A_{3,4,5,1} = \frac{2}{5}, \quad A_{\frac{3}{2},5,\frac{5}{2},1} = -\frac{1}{5}, \quad A_{2,6,3,1} = -\frac{1}{6},\\
A_{\frac{3}{2},6,\frac{5}{2},1} &= -\frac{2}{5}, \quad A_{\frac{3}{2},6,\frac{5}{2},2} = \frac{1}{30}, \quad A_{\frac{5}{2},4,\frac{7}{2},1} = \frac{2}{7}, \quad A_{\frac{5}{2},4,\frac{7}{2},2} = \frac{3}{56},\\
A_{\frac{5}{2},4,\frac{9}{2},1} &= \frac{1}{3}, \quad A_{\frac{5}{2},\frac{9}{2},4,1} = \frac{1}{4}, \quad A_{\frac{5}{2},\frac{9}{2},4,2} = \frac{1}{24}, \quad A_{\frac{5}{2},\frac{9}{2},5,1} = \frac{3}{10},\\
A_{3,4,4,1} &= \frac{3}{8}, \quad A_{3,4,4,2} = \frac{1}{12}, \quad A_{3,4,5,1} = \frac{2}{5}, \quad A_{3,\frac{9}{2},\frac{7}{2},1} = \frac{2}{7},\\
A_{3,\frac{9}{2},\frac{7}{2},2} &= \frac{3}{56}, \quad A_{3,\frac{9}{2},\frac{7}{2},3} = \frac{1}{126}, \quad A_{3,\frac{9}{2},\frac{9}{2},1} = \frac{1}{3}, \quad A_{3,\frac{9}{2},\frac{9}{2},2} = \frac{1}{15},\\
A_{3,\frac{9}{2},\frac{11}{2},1} &= \frac{4}{11}.
\end{aligned} \tag{A.2}$$

Appendix B The OPE between the stress energy tensor and the quasiprimary or primary fields in $c = 4$ model

In order to make sure whether a conformal field is quasi-primary field or not, one should compute the OPE between the stress energy tensor $T(z)$ and a field $\Phi(w)$ and check the vanishing of third-order pole in the OPE $T(z)\,\Phi(w)$. Now we list all the quasi-primary fields (where there exist three primary fields) in the sections 2 and 3

$$T(z)\,\left(TT-\frac{3}{10}\partial^2 T\right)(w)=\frac{1}{(z-w)^4}\,\frac{42}{5}\,T(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(GT-\frac{1}{8}\partial^2 G\right)(w)=\frac{1}{(z-w)^4}\,\frac{37}{8}\,G(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(G\partial T-\frac{4}{3}T\partial G+\frac{4}{15}\partial^3 G\right)(w)=\frac{1}{(z-w)^5}\,\frac{33}{5}\,G(w)$$
$$+\frac{1}{(z-w)^4}\,(-\frac{1}{3})\frac{33}{5}\,\partial G(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(GU-\frac{\sqrt{6}}{3}\partial W\right)(w)=\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(TW-\frac{3}{14}\partial^2 W\right)(w)=\frac{1}{(z-w)^4}\,\frac{71}{7}W(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(G\partial U-\frac{5}{3}\partial GU-\frac{\sqrt{6}}{7}\partial^2 W\right)(w)=-\frac{1}{(z-w)^4}\,\frac{124}{7}\,\sqrt{\frac{2}{3}}\,W(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(GW-\frac{1}{6\sqrt{6}}\partial^2 U\right)(w)=\frac{1}{(z-w)^4}\,5\sqrt{\frac{2}{3}}\,U(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(TU-\frac{1}{4}\partial^2 U\right)(w)=\frac{1}{(z-w)^4}\,\frac{33}{4}\,U(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(TO_{\frac{7}{2}}-\frac{3}{16}\partial^2 O_{\frac{7}{2}}\right)(w)=\frac{1}{(z-w)^4}\,\frac{193}{16}\,O_{\frac{7}{2}}(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(GP^{uu}_{4'}-\frac{4\sqrt{6}}{9}\partial O_{\frac{9}{2}}-\frac{\sqrt{6}}{56}\partial^2 O_{\frac{7}{2}}\right)(w)=\frac{1}{(z-w)^4}\,\frac{17}{4}\,\sqrt{\frac{3}{2}}\,O_{\frac{7}{2}}(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(GP^{ww}_{4'}-\frac{2}{9}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}}-\frac{1}{14}\sqrt{\frac{2}{3}}\partial^2 O_{\frac{7}{2}}\right)(w)=\frac{1}{(z-w)^4}\,\frac{17}{\sqrt{6}}\,O_{\frac{7}{2}}(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\partial P^{ww}_{4'}\right)(w)=+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\,\left(TP^{uu}_{4'}-\frac{1}{6}\partial^2 P^{uu}_{4'}\right)(w)=\frac{1}{(z-w)^4}\,14\,P^{uu}_{4'}(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(TP^{ww}_{4'}-\frac{1}{6}\partial^2 P^{ww}_{4'}\right)(w)=\frac{1}{(z-w)^4}\,14\,P^{ww}_{4'}(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(G\partial O_{\frac{7}{2}}-\frac{7}{3}\partial G O_{\frac{7}{2}}+\frac{1}{9\sqrt{6}}\partial^2 P^{uu}_{4'}-\frac{1}{3}\sqrt{\frac{2}{3}}\partial^2 P^{ww}_{4'}\right)(w)=$$

$$\frac{1}{(z-w)^4}\ \left[\frac{25}{3\sqrt{6}}\,P^{uu}_{4'}-25\sqrt{\frac{2}{3}}P^{ww}_{4'}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(GO_{\frac{9}{2}}-\frac{2}{63}\sqrt{\frac{2}{3}}\partial^2 P^{uu}_{4'}+\frac{1}{21\sqrt{6}}\partial^2 P^{ww}_{4'}\right)(w)=$$

$$\frac{1}{(z-w)^4}\ \left[\frac{104}{21}\sqrt{\frac{2}{3}}\,P^{uu}_{4'}-\frac{26}{7}\sqrt{\frac{2}{3}}P^{ww}_{4'}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(TO_{\frac{9}{2}}-\frac{3}{20}\partial^2 O_{\frac{9}{2}}\right)(w)=\frac{1}{(z-w)^4}\,\frac{319}{20}\,O_{\frac{9}{2}}(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(T\partial O_{\frac{7}{2}}-\frac{7}{4}\partial T O_{\frac{7}{2}}-\frac{1}{9}\partial^3 O_{\frac{7}{2}}\right)(w)=-\frac{1}{(z-w)^5}\,\frac{301}{12}O_{\frac{7}{2}}(w)$$

$$-\frac{1}{(z-w)^4}\,(-\frac{1}{7})\frac{301}{12}\,\partial O_{\frac{7}{2}}(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(G\partial P^{uu}_{4'}-\frac{8}{3}\partial G P^{uu}_{4'}-\frac{\sqrt{6}}{5}\partial^2 O_{\frac{9}{2}}-\frac{2}{63}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right)(w)=$$

$$-\frac{1}{(z-w)^5}\,\frac{50}{3}\,\sqrt{\frac{2}{3}}\,O_{\frac{7}{2}}(w)+\frac{1}{(z-w)^4}\,\left[-(-\frac{1}{7})\frac{50}{3}\sqrt{\frac{2}{3}}\,\partial O_{\frac{7}{2}}-\frac{286}{5}\sqrt{\frac{2}{3}}\,O_{\frac{9}{2}}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(G\partial P^{ww}_{4'}-\frac{8}{3}\partial G P^{ww}_{4'}-\frac{1}{5\sqrt{6}}\partial^2 O_{\frac{9}{2}}-\frac{8}{189}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right)(w)=$$

$$-\frac{1}{(z-w)^5}\,\frac{200}{9}\,\sqrt{\frac{2}{3}}\,O_{\frac{7}{2}}(w)+\frac{1}{(z-w)^4}\,\left[-(-\frac{1}{7})\frac{200}{9}\sqrt{\frac{2}{3}}\,\partial O_{\frac{7}{2}}-\frac{143}{15}\sqrt{\frac{2}{3}}\,O_{\frac{9}{2}}\right](w)$$

$$+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(T\partial U-\frac{5}{4}\partial T U-\frac{1}{7}\partial^3 U\right)(w)=-\frac{1}{(z-w)^5}\,\frac{345}{28}\,U(w)-\frac{1}{(z-w)^4}\,(-\frac{1}{5})\,\frac{345}{28}\,\partial U(w)$$

$$+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(G\partial W-2\partial G W-\frac{1}{21}\sqrt{\frac{2}{3}}\partial^3 U\right)(w)=-\frac{1}{(z-w)^5}\,\frac{55}{7}\sqrt{\frac{2}{3}}\,U(w)$$

$$-\frac{1}{(z-w)^4}\,(-\frac{1}{5})\frac{55}{7}\sqrt{\frac{2}{3}}\,\partial U(w)+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(GO_{4''}-\frac{2}{9}\partial O_{\frac{9}{2}'}\right)(w)=+\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(G\partial^2 U - 4\partial G \partial U + \frac{5}{2}\partial^2 G U - \frac{1}{2\sqrt{6}}\partial^3 W\right)(w) = \frac{1}{(z-w)^5}\, 9\sqrt{6}\, W(w)$$
$$+\frac{1}{(z-w)^4}\left[(-\frac{1}{6})9\sqrt{6}\partial W + \frac{75}{2}\left(GU - \frac{\sqrt{6}}{3}\partial W\right)\right](w) + \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(T\partial W - \frac{3}{2}\partial T W - \frac{1}{8}\partial^3 W\right)(w) = -\frac{1}{(z-w)^5} 18 W(w) - \frac{1}{(z-w)^4}(-\frac{1}{6}) 18 \partial W(w)$$
$$+\mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(T O_{4''} - \frac{1}{6}\partial^2 O_{4''}\right)(w) = \frac{1}{(z-w)^4}\, 14 O_{4''}(w) + \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(G O_{\frac{9}{2}'} - \frac{1}{9}\partial^2 O_{4''}\right)(w) = \frac{1}{(z-w)^4}\, \frac{52}{3} O_{4''}(w) + \mathcal{O}((z-w)^{-2}). \qquad \text{(B.1)}$$

One realizes that there is no third-order pole in (B.1). Using the definition of the coefficients (A.1), one can check those vanishing and nonvanishing coefficients appearing in (B.1) as follows:

$$\begin{aligned}
A_{2,4,2,1} &= 0, \quad A_{2,\frac{7}{2},\frac{3}{2},1} = 0, \quad A_{2,\frac{9}{2},\frac{3}{2},1} = -\frac{1}{3}, \quad A_{2,\frac{9}{2},\frac{3}{2},2} = 0, \\
A_{2,\frac{9}{2},\frac{5}{2},1} &= 0, \quad A_{2,5,3,1} = 0, \quad A_{2,\frac{11}{2},\frac{5}{2},1} = -\frac{1}{5}, \quad A_{2,\frac{11}{2},\frac{5}{2},2} = 0, \\
A_{2,\frac{11}{2},\frac{7}{2},1} &= 0, \quad A_{2,6,3,1} = -\frac{1}{6}, \quad A_{2,6,3,2} = 0, \quad A_{2,6,4,1} = 0, \\
A_{2,\frac{13}{2},\frac{7}{2},1} &= -\frac{1}{7}, \quad A_{2,\frac{13}{2},\frac{7}{2},2} = 0, \quad A_{2,\frac{13}{2},\frac{9}{2},1} = 0.
\end{aligned} \qquad \text{(B.2)}$$

For example, the first OPE in (B.1) has no third-order singular term. This can be realized by the vanishing of $A_{2,4,2,1}$ in (B.2). In other words, on the one hand, one can understand the vanishing of third-order singular term from the explicit WZW currents and on the other hand, one also checks this from the formula (1.3).

Note that one sees the following quasi-primary fields [44] from (B.1)

$$\left(T\Phi_i - \frac{3}{2(2h_i+1)}\partial^2\Phi_i\right)(z). \qquad \text{(B.3)}$$

The relative coefficient in (B.3) is fixed from the definition of quasi-primary condition. What about other quasi-primary fields that contain the derivatives in the quadratic normal ordered product?

It is known, in [15], that any quasi-primary field can be written in terms of quadratic part and linear part

$$\sum_{r=0}^{n} B_{i,j,n,r}\, \partial^r\, N(\Phi_j, \partial^{n-r}\Phi_i) + \sum_{k:\, h_i+h_j-h_k \geq 1} C_{ijk}\, C_{i,j,k,n}\, \partial^{h_i+h_j-h_k+n}\Phi_k. \qquad \text{(B.4)}$$

When $n = 0$, the first term in (B.4) does not contain any derivatives. We introduce each coefficient functions

$$B_{i,j,n,r} \equiv (-1)^r \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)} \frac{\Gamma(2h_i+n)}{\Gamma(r+1)\Gamma(2h_i+n-r)} \frac{\Gamma(r+1)\Gamma(2h_i+2h_j+2n-r-1)}{\Gamma(2h_i+2h_j+2n-1)},$$
$$C_{i,j,k,n} \equiv -(-1)^n \frac{\Gamma(h_i+h_j-h_k+n)}{\Gamma(h_i+h_j-h_k)\Gamma(n+1)} \frac{\Gamma(n+1)\Gamma(2h_i+2h_j+n-1)}{\Gamma(2h_i+2h_j+2n-1)}$$
$$\times \frac{\Gamma(2h_i+n)}{\Gamma(h_i+h_j-h_k+n+1)\Gamma(h_i-h_j+h_k)} \frac{\Gamma(h_i+h_j-h_k)\Gamma(2h_k)}{\Gamma(h_i+h_j+h_k-1)} \frac{1}{(h_i+h_j+h_k+n-1)}$$
$$\times \frac{1}{\Gamma(h_i+h_j-h_k)}. \tag{B.5}$$

We simply rewrite their binomial symbols as the Gamma functions. Although one can simplify $B_{i,j,n,r}$ and $C_{i,j,k,n}$ further, one keeps their forms in its present form. The structure constant C_{ijk} is the same as the one in (1.3) where the fields $\Phi_i(z)$ and $\Phi_j(w)$ appear in the first quadratic part in (B.4). This implies that the complete structure of quasi-primary field is determined only if one knows the OPE $\Phi_i(z)\ \Phi_j(w)$ because the second term contains the above structure constant C_{ijk}. Otherwise, this structure constant is not known and when one uses the Jacobi identities mentioned before, this unknown structure constant is fixed by the Jacobi identities.

Let us emphasize that their normal ordered product is different from the one [16] we use in this paper. In other words [45], recall that

$$N(\Phi_j, \partial^{n-r}\Phi_i)(z) = (\partial^{n-r}\Phi_i\, \Phi_j)(z). \tag{B.6}$$

Once one uses their formula (B.4), one should also apply the convention (B.6) to the expression of quasi-primary field. It is better to take one example. In (B.1), one sees the quasi-primary field $\left(G\partial T - \frac{4}{3}T\partial G + \frac{4}{15}\partial^3 G\right)(z)$. How one can observe this from the formula (B.4)? Now let us take $\Phi_i(z) = T(z), \Phi_j(z) = G(z)$. Then one has $T(z)\ G(w) = \frac{1}{(z-w)^2}\frac{3}{2}G(w) + \frac{1}{(z-w)}\partial G(w) + \cdots$. From this, one can identify the field $\Phi_k(z) = G(z)$ with $C_{ijk} = \frac{3}{2}$. From the expression (B.5), one has $B_{2,\frac{3}{2},1,0} = 1, B_{2,\frac{3}{2},1,1} = -\frac{4}{7}$ and $C_{2,\frac{3}{2},\frac{3}{2},1} = \frac{8}{105}$. Then by substitiuting these numerical values in (B.4), one obtains $\frac{3}{7}N(G\partial T)(z) - \frac{4}{7}N(\partial GT)(z) + \frac{4}{35}\partial^3 G(z)$. However, due to the fact (B.6), one has $\frac{3}{7}\left(\partial TG - \frac{4}{3}T\partial G + \frac{4}{15}\partial^3 G\right)(z)$. Furthermore, by using the relation $G\partial T(z) = \partial TG(z)$, one finally obtains $\frac{3}{7}\left(G\partial T - \frac{4}{3}T\partial G + \frac{4}{15}\partial^3 G\right)(z)$ which is proportional to the quasi-primary field one mentioned before. One can do similar analysis and checks that all the quasi-primary fields appearing in this paper can be read off from the general formula.

One can check all the quasi-primary fields by using the formula (B.4). We list the relevant coefficient functions which appear in the quasi-primary fields in (B.1):

$$B_{2,\frac{3}{2},1,0} = 1, \qquad B_{2,\frac{3}{2},1,1} = -\frac{4}{7}, \qquad B_{\frac{5}{2},\frac{3}{2},1,0} = 1, \qquad B_{\frac{5}{2},\frac{3}{2},1,1} = -\frac{5}{8},$$

$$
\begin{aligned}
B_{\frac{7}{2},\frac{3}{2},1,0} &= 1, \quad B_{\frac{7}{2},\frac{3}{2},1,1} = -\frac{7}{10}, \quad B_{\frac{7}{2},2,1,0} = 1, \quad B_{\frac{7}{2},2,1,1} = -\frac{7}{11}, \\
B_{4,\frac{3}{2},1,0} &= 1, \quad B_{4,\frac{3}{2},1,1} = -\frac{8}{11}, \quad B_{\frac{5}{2},2,1,0} = 1, \quad B_{\frac{5}{2},2,1,1} = -\frac{5}{9}, \\
B_{3,\frac{3}{2},1,0} &= 1, \quad B_{3,\frac{3}{2},1,1} = -\frac{2}{3}, \quad B_{\frac{5}{2},\frac{3}{2},2,0} = 1, \quad B_{\frac{5}{2},\frac{3}{2},2,1} = -\frac{6}{5}, \\
B_{\frac{5}{2},\frac{3}{2},2,2} &= \frac{1}{3}, \quad B_{3,2,1,0} = 1, \quad B_{3,2,1,1} = -\frac{3}{5}, \quad B_{\frac{3}{2},\frac{5}{2},0,0} = 1, \\
B_{\frac{3}{2},3,0,0} &= 1, \quad B_{\frac{3}{2},4,0,0} = 1, \quad B_{\frac{3}{2},\frac{7}{2},0,0} = 1, \quad B_{\frac{3}{2},\frac{9}{2},0,0} = 1,
\end{aligned}
\tag{B.7}
$$

and

$$
\begin{aligned}
C_{2,\frac{3}{2},\frac{3}{2},1} &= \frac{8}{105}, \quad C_{\frac{5}{2},\frac{3}{2},3,1} = \frac{5}{28}, \quad C_{\frac{7}{2},\frac{3}{2},4,1} = \frac{7}{30}, \quad C_{\frac{7}{2},2,\frac{7}{2},1} = \frac{10}{99}, \\
C_{4,\frac{3}{2},\frac{7}{2},1} &= \frac{16}{99}, \quad C_{4,\frac{3}{2},\frac{9}{2},1} = \frac{14}{55}, \quad C_{\frac{5}{2},2,\frac{5}{2},1} = \frac{4}{63}, \quad C_{3,\frac{3}{2},\frac{5}{2},1} = \frac{8}{63}, \\
C_{\frac{5}{2},\frac{3}{2},3,2} &= -\frac{1}{18}, \quad C_{3,2,3,1} = \frac{1}{12}, \quad C_{\frac{3}{2},\frac{5}{2},3,0} = -\frac{1}{3}, \quad C_{\frac{3}{2},3,\frac{5}{2},0} = -\frac{1}{30}, \\
C_{\frac{3}{2},4,\frac{7}{2},0} &= -\frac{1}{56}, \quad C_{\frac{3}{2},4,\frac{9}{2},0} = -\frac{2}{9}, \quad C_{\frac{3}{2},\frac{7}{2},4,0} = -\frac{1}{4}, \quad C_{\frac{3}{2},\frac{9}{2},4,0} = -\frac{1}{72}.
\end{aligned}
\tag{B.8}
$$

Due to the fact that $B_{2,i,0,0} = 1$, $C_{2,i,i,0} = -\frac{3}{2h_i(2h_i+1)}$ and $C_{2ii} = h_i$(the second-order pole of OPE $\Phi_i(z)\ T(w)$ is given by $\frac{1}{(z-w)^2} h_i \Phi_i(w)$ and there is also first-order singular term), one observes the previous relation (B.3).

What happens when the spin-$\frac{3}{2}$ current $G(z)$ is combined with any primary field $\Phi_i(z)$ of spin-h_i? As done in (B.7) and (B.8), one computes $B_{\frac{3}{2},i,0,0} = 1$, $C_{\frac{3}{2},i,i-\frac{1}{2},0} = -\frac{1}{4h_i(h_i-\frac{1}{2})}$ and $C_{2,i,i-\frac{1}{2}} = 2(h_i - \frac{1}{2})$(when the second-order pole of OPE $\Phi_i(z)\ G(w)$ is given by $\frac{1}{(z-w)^2} 2(h_i - \frac{1}{2})\Phi_{i-\frac{1}{2}}(w)$ and there exists a first-order singular term), one observes similar relation to (B.3). Furthermore, when the first-order pole of OPE $\Phi_i(z)\ G(w)$ is given by $\frac{1}{(z-w)} \Phi_{i+\frac{1}{2}}(w)$, then $C_{2,i,i+\frac{1}{2}} = 1$ with $B_{\frac{3}{2},i,0,0} = 1$ and $C_{\frac{3}{2},i,i+\frac{1}{2},0} = -\frac{1}{(h_i+\frac{1}{2})}$. From these two cases, one arrives at

$$
\left(G\Phi_i - \frac{1}{2h_i} \partial^2 \Phi_{i-\frac{1}{2}} \right)(z), \qquad \text{or} \qquad \left(G\Phi_i - \frac{1}{(h_i + \frac{1}{2})} \partial \Phi_{i+\frac{1}{2}} \right)(z).
\tag{B.9}
$$

Therefore, one realizes that all the quasi-primary fields containing $T(z)$ or $G(z)$ in this paper can be classified by (B.3) and (B.9).

For the case $\Phi_i(z) = P^{uu}_{4'}(z)$ or $\Phi_i(z) = P^{ww}_{4'}(z)$, one should use the mixed form of (B.9). Note that the explicit OPEs are given in the footnote 9. When the field $\Phi_i(z)$ or $\Phi_j(w)$ does not contain $T(z)$ or $G(z)$, then one should use the original expression (B.5). This will occur when we compute the OPEs between the higher spin currents we did not consider.

Appendix C The OPE between the spin-$\frac{3}{2}$ current and the quasiprimary or primary field fields in $c = 4$ model

For the correct superpartner in the given superfield, one should know the OPE between the spin-$\frac{3}{2}$ current and the arbitrary quasi-primary fields. We present them here

$$
\begin{aligned}
& G(z)\ \left(TT - \frac{3}{10}\partial^2 T\right)(w) = \frac{1}{(z-w)^4}\,\frac{51}{20}\,G(w) + \frac{1}{(z-w)^3}\,(-\frac{1}{3})\frac{51}{20}\partial G(w) + \mathcal{O}((z-w)^{-2}), \\
& G(z)\ \left(GT - \frac{1}{8}\partial^2 G\right)(w) = \frac{1}{(z-w)^3}\,\frac{37}{6}\,T(w) + \mathcal{O}((z-w)^{-2}), \\
& G(z)\ \left(G\partial T - \frac{4}{3}T\partial G + \frac{4}{15}\partial^3 G\right)(w) = -\frac{1}{(z-w)^4}\,\frac{44}{5}\,T(w) \\
& -\frac{1}{(z-w)^3}\,(-\frac{1}{4})\frac{44}{5}\,\partial T(w) + \mathcal{O}((z-w)^{-2}), \\
& G(z)\ \left(GU - \frac{\sqrt{6}}{3}\partial W\right)(w) = \frac{1}{(z-w)^3}\,\frac{13}{3}\,U(w) + \mathcal{O}((z-w)^{-2}), \\
& G(z)\ \left(TW - \frac{3}{14}\partial^2 W\right)(w) = \frac{1}{(z-w)^4}\,\frac{155}{14\sqrt{6}}\,U(w) \\
& +\frac{1}{(z-w)^3}\,(-\frac{1}{5})\frac{155}{14\sqrt{6}}\partial U(w) + \mathcal{O}((z-w)^{-2}), \\
& G(z)\ \left(G\partial U - \frac{5}{3}\partial GU - \frac{\sqrt{6}}{7}\partial^2 W\right)(w) = -\frac{1}{(z-w)^4}\,\frac{335}{21}\,U(w) \\
& -\frac{1}{(z-w)^3}\,(-\frac{1}{5})\frac{335}{21}\partial U(w) + \mathcal{O}((z-w)^{-2}), \\
& G(z)\ \left(GW - \frac{1}{6\sqrt{6}}\partial^2 U\right)(w) = \frac{1}{(z-w)^3}\,\frac{25}{3}W(w) + \mathcal{O}((z-w)^{-2}), \\
& G(z)\ \left(TU - \frac{1}{4}\partial^2 U\right)(w) = \frac{1}{(z-w)^3}\,2\sqrt{6}W(w) + \mathcal{O}((z-w)^{-2}), \\
& G(z)\ \left(TO_{\frac{7}{2}} - \frac{3}{16}\partial^2 O_{\frac{7}{2}}\right)(w) = \frac{1}{(z-w)^3}\,\left[-\frac{17}{8\sqrt{6}}P^{uu}_{4'} + \frac{17}{4}\sqrt{\frac{3}{2}}P^{ww}_{4'}\right](w) + \mathcal{O}((z-w)^{-2}), \\
& G(z)\ \left(GP^{uu}_{4'} - \frac{4\sqrt{6}}{9}\partial O_{\frac{9}{2}} - \frac{\sqrt{6}}{56}\partial^2 O_{\frac{7}{2}}\right)(w) = \\
& \frac{1}{(z-w)^3}\,\left[\frac{239}{36}P^{uu}_{4'} + \frac{17}{6}P^{ww}_{4'}\right](w) + \mathcal{O}((z-w)^{-2}), \\
& G(z)\ \left(GP^{ww}_{4'} - \frac{2}{9}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}} - \frac{1}{14}\sqrt{\frac{2}{3}}\partial^2 O_{\frac{7}{2}}\right)(w) =
\end{aligned}
$$

$$\frac{1}{(z-w)^3}\left[-\frac{17}{27}P^{uu}_{4'}+\frac{98}{9}P^{ww}_{4'}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\partial P^{ww}_{4'}\right)(w)=\frac{1}{(z-w)^3}\,\frac{37}{6}\,O_{\frac{7}{2}}(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\left(TP^{uu}_{4'}-\frac{1}{6}\partial^2 P^{uu}_{4'}\right)(w)=\frac{1}{(z-w)^4}\,5\sqrt{\frac{3}{2}}\,O_{\frac{7}{2}}(w)$$

$$+\frac{1}{(z-w)^3}\left[(-\frac{1}{7})5\sqrt{\frac{3}{2}}\,\partial O_{\frac{7}{2}}+13\sqrt{\frac{2}{3}}O_{\frac{9}{2}}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\left(TP^{ww}_{4'}-\frac{1}{6}\partial^2 P^{ww}_{4'}\right)(w)=\frac{1}{(z-w)^4}\,10\sqrt{\frac{3}{2}}\,O_{\frac{7}{2}}(w)$$

$$+\frac{1}{(z-w)^3}\left[(-\frac{1}{7})10\sqrt{\frac{3}{2}}\,\partial O_{\frac{7}{2}}+\frac{13}{3\sqrt{6}}O_{\frac{9}{2}}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\left(G\partial O_{\frac{7}{2}}-\frac{7}{3}\partial GO_{\frac{7}{2}}+\frac{1}{9\sqrt{6}}\partial^2 P^{uu}_{4'}-\frac{1}{3}\sqrt{\frac{2}{3}}\partial^2 P^{ww}_{4'}\right)(w)=$$

$$-\frac{1}{(z-w)^4}\,\frac{77}{3}\,O_{\frac{7}{2}}(w)-\frac{1}{(z-w)^3}\,(-\frac{1}{7})\frac{77}{3}\,\partial O_{\frac{7}{2}}(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\left(GO_{\frac{9}{2}}-\frac{2}{63}\sqrt{\frac{2}{3}}\partial^2 P^{uu}_{4'}+\frac{1}{21\sqrt{6}}\partial^2 P^{ww}_{4'}\right)(w)=\frac{1}{(z-w)^3}\,\frac{103}{9}\,O_{\frac{9}{2}}(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\left(TO_{\frac{9}{2}}-\frac{3}{20}\partial^2 O_{\frac{9}{2}}\right)(w)=\frac{1}{(z-w)^4}\left[\frac{208}{35}\sqrt{\frac{2}{3}}P^{uu}_{4'}-\frac{52}{35}\sqrt{6}\,P^{ww}_{4'}\right](w)$$

$$+\frac{1}{(z-w)^3}\,(-\frac{1}{8})\partial\left[\frac{208}{35}\sqrt{\frac{2}{3}}P^{uu}_{4'}-\frac{52}{35}\sqrt{6}\,P^{ww}_{4'}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\left(T\partial O_{\frac{7}{2}}-\frac{7}{4}\partial TO_{\frac{7}{2}}-\frac{1}{9}\partial^3 O_{\frac{7}{2}}\right)(w)=\frac{1}{(z-w)^4}\left[\frac{25}{6\sqrt{6}}\,P^{uu}_{4'}-\frac{25}{\sqrt{6}}\,P^{ww}_{4'}\right](w)$$

$$+\frac{1}{(z-w)^3}\left[(-\frac{1}{8})\frac{25}{6\sqrt{6}}\,\partial P^{uu}_{4'}-(-\frac{1}{8})\frac{25}{\sqrt{6}}\,\partial P^{ww}_{4'}\right.$$

$$\left.-\frac{21}{4}\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\,\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\,\partial P^{ww}_{4'}\right)\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\left(G\partial P^{uu}_{4'}-\frac{8}{3}\partial GP^{uu}_{4'}-\frac{\sqrt{6}}{5}\partial^2 O_{\frac{9}{2}}-\frac{2}{63}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right)(w)=$$

$$\frac{1}{(z-w)^4}\left[-\frac{1444}{45}\,P^{uu}_{4'}+\frac{56}{15}\,P^{ww}_{4'}\right](w)+\frac{1}{(z-w)^3}\left[-(-\frac{1}{8})\frac{1444}{45}\,\partial P^{uu}_{4'}+(-\frac{1}{8})\frac{56}{15}\,\partial P^{ww}_{4'}\right.$$

$$\left.-2\sqrt{6}\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\,\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\,\partial P^{ww}_{4'}\right)\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$
\begin{aligned}
&G(z)\ \left(G\partial P^{ww}_{4'}-\frac{8}{3}\partial G P^{ww}_{4'}-\frac{1}{5\sqrt{6}}\partial^2 O_{\frac{9}{2}}-\frac{8}{189}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right)(w)=\\
&\frac{1}{(z-w)^4}\ \left[-\frac{112}{135}\,P^{uu}_{4'}-\frac{1192}{45}\,P^{ww}_{4'}\right](w)+\frac{1}{(z-w)^3}\ \left[-(-\frac{1}{8})\frac{112}{135}\,\partial P^{uu}_{4'}-(-\frac{1}{8})\frac{1192}{45}\ \partial P^{ww}_{4'}\right.\\
&\left.-8\sqrt{\frac{2}{3}}\ \left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\,\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\ \partial P^{ww}_{4'}\right)\right](w)+\mathcal{O}((z-w)^{-2}),\\
&G(z)\ \left(T\partial U-\frac{5}{4}\partial TU-\frac{1}{7}\partial^3 U\right)(w)=-\frac{1}{(z-w)^4}\,\frac{33}{7}\sqrt{\frac{3}{2}}\,W(w)\\
&+\frac{1}{(z-w)^3}\left[-(-\frac{1}{6})\frac{33}{7}\sqrt{\frac{3}{2}}\partial W-\frac{15}{4}\left(GU-\frac{\sqrt{6}}{3}\partial W\right)\right](w)+\mathcal{O}((z-w)^{-2}),\\
&G(z)\ \left(G\partial W-2\partial GW-\frac{1}{21}\sqrt{\frac{2}{3}}\partial^3 U\right)(w)=-\frac{1}{(z-w)^4}\,\frac{116}{7}\,W(w)\\
&+\frac{1}{(z-w)^3}\left[-(-\frac{1}{6})\frac{116}{7}\partial W-5\sqrt{\frac{2}{3}}\left(GU-\frac{\sqrt{6}}{3}\partial W\right)\right](w)+\mathcal{O}((z-w)^{-2}),\\
&G(z)\ \left(GO_{4''}-\frac{2}{9}\partial O_{\frac{9}{2}'}\right)(w)=\frac{1}{(z-w)^3}\frac{64}{9}O_{4''}(w)+\mathcal{O}((z-w)^{-2}),\\
&G(z)\ \left(G\partial^2 U-4\partial G\partial U+\frac{5}{2}\partial^2 GU-\frac{1}{2\sqrt{6}}\partial^3 W\right)(w)=\frac{1}{(z-w)^5}\,85\,U(w)\\
&+\frac{1}{(z-w)^4}(-\frac{2}{5})85\partial U(w)\\
&+\frac{1}{(z-w)^3}\left[(\frac{1}{30})85\partial^2 U-2\sqrt{6}\left(GW-\frac{1}{6\sqrt{6}}\partial^2 U\right)+10\left(TU-\frac{1}{4}\partial^2 U\right)\right](w)+\mathcal{O}((z-w)^{-2}),\\
&G(z)\ \left(T\partial W-\frac{3}{2}\partial TW-\frac{1}{8}\partial^3 W\right)(w)=-\frac{1}{(z-w)^5}\,\frac{5}{2}\sqrt{\frac{3}{2}}\,U(w)-\frac{1}{(z-w)^4}(-\frac{2}{5})\frac{5}{2}\sqrt{\frac{3}{2}}\partial U(w)\\
&+\frac{1}{(z-w)^3}\left[-(\frac{1}{30})\frac{5}{2}\sqrt{\frac{3}{2}}\partial^2 U-\frac{9}{2}\left(GW-\frac{1}{6\sqrt{6}}\partial^2 U\right)+5\sqrt{\frac{2}{3}}\left(TU-\frac{1}{4}\partial^2 U\right)\right](w)\\
&+\mathcal{O}((z-w)^{-2}),\\
&G(z)\ \left(TO_{4''}-\frac{1}{6}\partial^2 O_{4''}\right)(w)=\frac{1}{(z-w)^3}\,\frac{13}{6}O_{\frac{9}{2}'}(w)+\mathcal{O}((z-w)^{-2}),\\
&G(z)\ \left(GO_{\frac{9}{2}'}-\frac{1}{9}\partial^2 O_{4''}\right)(w)=\frac{1}{(z-w)^3}\,\frac{103}{9}O_{\frac{9}{2}'}(w)+\mathcal{O}((z-w)^{-2}).
\end{aligned}
\tag{C.1}
$$

Using the definition of the coefficients (A.1), one can check those vanishing and nonvanishing coefficients appearing in (C.1) as follows:

$$
A_{\frac{3}{2},4,\frac{3}{2},1} = -\frac{1}{3}, \qquad A_{\frac{3}{2},4,\frac{3}{2},2}=0, \qquad A_{\frac{3}{2},\frac{9}{2},2,1}=-\frac{1}{4}, \qquad A_{\frac{3}{2},5,\frac{5}{2},1}=-\frac{1}{5},
$$

$$A_{\frac{3}{2},\frac{11}{2},3,1} = -\frac{1}{6}, \qquad A_{\frac{3}{2},6,\frac{5}{2},1} = -\frac{2}{5}, \qquad A_{\frac{3}{2},6,\frac{5}{2},2} = \frac{1}{30}, \qquad A_{\frac{3}{2},6,\frac{7}{2},1} = -\frac{1}{7},$$
$$A_{\frac{3}{2},\frac{13}{2},4,1} = -\frac{1}{8}. \tag{C.2}$$

For example, the first OPE in (C.1) has the third-order singular term with the coefficient $-\frac{1}{3}$ which coincides with the value $A_{\frac{3}{2},4,\frac{3}{2},1} = -\frac{1}{3}$.

Appendix D The OPE between the stress energy tensor and the quasi-primary or primary fields in the $c < 4$ coset model

In previous Appendices B and C, the central charge c was fixed as $c = 4$. Now we consider the OPEs for general central charge $c < 4$. The OPEs are given by

$$T(z)\ \left(TT - \frac{3}{10}\partial^2 T\right)(w) = \frac{1}{(z-w)^4}\left[\frac{6(66+63k+7k^2)}{5(3+k)(6+k)}\right] T(w) + \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(GT - \frac{1}{8}\partial^2 G\right)(w) = \frac{1}{(z-w)^4}\left[\frac{378+333k+37k^2}{8(3+k)(6+k)}\right] G(w) + \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(G\partial T - \frac{4}{3}T\partial G + \frac{4}{15}\partial^3 G\right)(w) = \frac{1}{(z-w)^5}\left[\frac{3(-42+99k+11k^2)}{5(3+k)(6+k)}\right] G(w)$$
$$+\frac{1}{(z-w)^4}\,(-\frac{1}{3})\left[\frac{3(-42+99k+11k^2)}{5(3+k)(6+k)}\right]\partial G(w) + \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(GU - \frac{\sqrt{6}}{3}\partial W\right)(w) = \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(TW - \frac{3}{14}\partial^2 W\right)(w) = \frac{1}{(z-w)^4}\left[\frac{1026+639k+71k^2}{7(3+k)(6+k)}\right] W(w) + \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(G\partial U - \frac{5}{3}\partial G U - \frac{\sqrt{6}}{7}\partial^2 W\right)(w) = -\frac{1}{(z-w)^4}\,\frac{124}{7}\sqrt{\frac{2}{3}}\,W(w) + \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(GW - \frac{1}{6\sqrt{6}}\partial^2 U\right)(w) = \frac{1}{(z-w)^4}\,5\sqrt{\frac{2}{3}}\,U(w) + \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(TU - \frac{1}{4}\partial^2 U\right)(w) = \frac{1}{(z-w)^4}\left[\frac{3(150+99k+11k^2)}{4(3+k)(6+k)}\right] U(w) + \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(TO_{\frac{7}{2}} - \frac{3}{16}\partial^2 O_{\frac{7}{2}}\right)(w) = \frac{1}{(z-w)^4}\left[\frac{2898+1737k+193k^2}{16(3+k)(6+k)}\right] O_{\frac{7}{2}}(w) + \mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(GP^{uu}_{4'} - \frac{4\sqrt{6}}{9}\partial O_{\frac{9}{2}} - \frac{\sqrt{6}}{56}\partial^2 O_{\frac{7}{2}}\right)(w) = \frac{1}{(z-w)^4}\,\frac{17}{4}\sqrt{\frac{3}{2}}\,O_{\frac{7}{2}}(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(GP^{ww}_{4'} - \frac{2}{9}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}} - \frac{1}{14}\sqrt{\frac{2}{3}}\partial^2 O_{\frac{7}{2}}\right)(w) = \frac{1}{(z-w)^4}\,\frac{17}{\sqrt{6}}\,O_{\frac{7}{2}}(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(GO_{\frac{7}{2}} + \frac{1}{4\sqrt{6}}\partial P^{uu}_{4'} - \frac{\sqrt{6}}{4}\partial P^{ww}_{4'}\right)(w) = +\mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(TP^{uu}_{4'} - \frac{1}{6}\partial^2 P^{uu}_{4'}\right)(w) = \frac{1}{(z-w)^4}\left[\frac{2(108+63k+7k^2)}{(3+k)(6+k)}\right]P^{uu}_{4'}(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(TP^{ww}_{4'} - \frac{1}{6}\partial^2 P^{ww}_{4'}\right)(w) = \frac{1}{(z-w)^4}\left[\frac{2(108+63k+7k^2)}{(3+k)(6+k)}\right]P^{ww}_{4'}(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(G\partial O_{\frac{7}{2}} - \frac{7}{3}\partial GO_{\frac{7}{2}} + \frac{1}{9\sqrt{6}}\partial^2 P^{uu}_{4'} - \frac{1}{3}\sqrt{\frac{2}{3}}\partial^2 P^{ww}_{4'}\right)(w) =$$

$$\frac{1}{(z-w)^4}\ \left[\frac{25}{3\sqrt{6}}\,P^{uu}_{4'} - 25\sqrt{\frac{2}{3}}P^{ww}_{4'}\right](w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(GO_{\frac{9}{2}} - \frac{2}{63}\sqrt{\frac{2}{3}}\partial^2 P^{uu}_{4'} + \frac{1}{21\sqrt{6}}\partial^2 P^{ww}_{4'}\right)(w) =$$

$$\frac{1}{(z-w)^4}\ \left[\frac{104}{21}\sqrt{\frac{2}{3}}\,P^{uu}_{4'} - \frac{26}{7}\sqrt{\frac{2}{3}}P^{ww}_{4'}\right](w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(TO_{\frac{9}{2}} - \frac{3}{20}\partial^2 O_{\frac{9}{2}}\right)(w) = \frac{1}{(z-w)^4}\left[\frac{(5022+2871k+319k^2)}{20(3+k)(6+k)}\right]O_{\frac{9}{2}}(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(T\partial O_{\frac{7}{2}} - \frac{7}{4}\partial TO_{\frac{7}{2}} - \frac{1}{9}\partial^3 O_{\frac{7}{2}}\right)(w) = -\frac{1}{(z-w)^5}\left[\frac{7(342+387k+43k^2)}{12(3+k)(6+k)}\right]O_{\frac{7}{2}}(w)$$

$$-\frac{1}{(z-w)^4}\,(-\frac{1}{7})\left[\frac{7(342+387k+43k^2)}{12(3+k)(6+k)}\right]\partial O_{\frac{7}{2}}(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(G\partial P^{uu}_{4'} - \frac{8}{3}\partial GP^{uu}_{4'} - \frac{\sqrt{6}}{5}\partial^2 O_{\frac{9}{2}} - \frac{2}{63}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right)(w) =$$

$$-\frac{1}{(z-w)^5}\,\frac{50}{3}\,\sqrt{\frac{2}{3}}\,O_{\frac{7}{2}}(w) + \frac{1}{(z-w)^4}\left[-(-\frac{1}{7})\frac{50}{3}\sqrt{\frac{2}{3}}\,\partial O_{\frac{7}{2}} - \frac{286}{5}\sqrt{\frac{2}{3}}\,O_{\frac{9}{2}}\right](w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(G\partial P^{ww}_{4'} - \frac{8}{3}\partial GP^{ww}_{4'} - \frac{1}{5\sqrt{6}}\partial^2 O_{\frac{9}{2}} - \frac{8}{189}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right)(w) =$$

$$-\frac{1}{(z-w)^5}\,\frac{200}{9}\,\sqrt{\frac{2}{3}}\,O_{\frac{7}{2}}(w) + \frac{1}{(z-w)^4}\left[-(-\frac{1}{7})\frac{200}{9}\sqrt{\frac{2}{3}}\,\partial O_{\frac{7}{2}} - \frac{143}{15}\sqrt{\frac{2}{3}}\,O_{\frac{9}{2}}\right](w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(T\partial U - \frac{5}{4}\partial TU - \frac{1}{7}\partial^3 U\right)(w) = -\frac{1}{(z-w)^5}\left[\frac{15(78+207k+23k^2)}{28(3+k)(6+k)}\right]U(w)$$

$$-\frac{1}{(z-w)^4}\,(-\frac{1}{5})\left[\frac{15(78+207k+23k^2)}{28(3+k)(6+k)}\right]\partial U(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z)\ \left(G\partial W-2\partial GW-\frac{1}{21}\sqrt{\frac{2}{3}}\partial^3 U\right)(w)=-\frac{1}{(z-w)^5}\,\frac{55}{7}\sqrt{\frac{2}{3}}\,U(w)$$
$$-\frac{1}{(z-w)^4}\,(-\frac{1}{5})\,\partial U(w)+\mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(GO_{4''}-\frac{2}{9}\partial O_{\frac{9}{2}'}\right)(w)=+\mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(G\partial^2 U-4\partial G\partial U+\frac{5}{2}\partial^2 GU-\frac{1}{2\sqrt{6}}\partial^3 W\right)(w)=\frac{1}{(z-w)^5}\,9\sqrt{6}\,W(w)$$
$$+\frac{1}{(z-w)^4}\left[-3\sqrt{\frac{3}{2}}\partial W+\frac{75}{2}\left(GU-\frac{\sqrt{6}}{3}\partial W\right)\right](w)+\mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(T\partial W-\frac{3}{2}\partial TW-\frac{1}{8}\partial^3 W\right)(w)=-\frac{1}{(z-w)^5}\left[\frac{18(6+9k+k^2)}{(3+k)(6+k)}\right]W(w)$$
$$-\frac{1}{(z-w)^4}(-\frac{1}{6})\left[\frac{18(6+9k+k^2)}{(3+k)(6+k)}\right]\partial W(w)+\mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(TO_{4''}-\frac{1}{6}\partial^2 O_{4''}\right)(w)=\frac{1}{(z-w)^4}\left[\frac{2(108+63k+7k^2)}{(3+k)(6+k)}\right]O_{4''}(w)+\mathcal{O}((z-w)^{-2}),$$
$$T(z)\ \left(GO_{\frac{9}{2}'}-\frac{1}{9}\partial^2 O_{4''}\right)(w)=\frac{1}{(z-w)^4}\,\frac{52}{3}O_{4''}(w)+\mathcal{O}((z-w)^{-2}). \tag{D.1}$$

Of course, after the $k\to\infty$ limit, the above OPEs (D.1) become the OPEs (B.1). One sees the k-dependence in the quasi-primary fields which contain the $T(w)$ because the OPE between $T(z)$ and $T(w)$ contains the central charge. On the other hand, the OPEs between the $T(z)$ and the quasi-primary fields which do not have $T(w)$ in their expression are the same as the ones for $c=4$ model. In this case, the previous coefficients (B.2) hold.

Appendix E The OPE between the spin-$\frac{3}{2}$ current and the quasi-primary or primary fields in the $c<4$ coset model

Similarly, one can proceed for the general central charge and the OPEs are given by

$$G(z)\ \left(TT-\frac{3}{10}\partial^2 T\right)(w)=\frac{1}{(z-w)^4}\,\frac{51}{20}\,G(w)+\frac{1}{(z-w)^3}\,(-\frac{1}{3})\frac{51}{20}\partial G(w)+\mathcal{O}((z-w)^{-2}),$$
$$G(z)\ \left(GT-\frac{1}{8}\partial^2 G\right)(w)=\frac{1}{(z-w)^3}\left[\frac{378+333k+37k^2}{6(3+k)(6+k)}\right]T(w)+\mathcal{O}((z-w)^{-2}),$$
$$G(z)\ \left(G\partial T-\frac{4}{3}T\partial G+\frac{4}{15}\partial^3 G\right)(w)=-\frac{1}{(z-w)^4}\left[\frac{4(-42+99k+11k^2)}{5(3+k)(6+k)}\right]T(w)$$

$$-\frac{1}{(z-w)^3}\,(-\frac{1}{4})\left[\frac{4(-42+99k+11k^2)}{5(3+k)(6+k)}\right]\partial T(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(GU-\frac{\sqrt{6}}{3}\partial W\right)(w)=\frac{1}{(z-w)^3}\left[\frac{90+117k+13k^2}{3(3+k)(6+k)}\right]U(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(TW-\frac{3}{14}\partial^2 W\right)(w)=\frac{1}{(z-w)^4}\,\frac{155}{14\sqrt{6}}\,U(w)$$

$$+\frac{1}{(z-w)^3}\,(-\frac{1}{5})\frac{155}{14\sqrt{6}}\partial U(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(G\partial U-\frac{5}{3}\partial GU-\frac{\sqrt{6}}{7}\partial^2 W\right)(w)=-\frac{1}{(z-w)^4}\left[\frac{5(198+603k+67k^2)}{21(3+k)(6+k)}\right]U(w)$$

$$-\frac{1}{(z-w)^3}\,(-\frac{1}{5})\left[\frac{5(198+603k+67k^2)}{21(3+k)(6+k)}\right]\partial U(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(GW-\frac{1}{6\sqrt{6}}\partial^2 U\right)(w)=\frac{1}{(z-w)^3}\left[\frac{306+225k+25k^2}{3(3+k)(6+k)}\right]W(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(TU-\frac{1}{4}\partial^2 U\right)(w)=\frac{1}{(z-w)^3}\,2\sqrt{6}W(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(TO_{\frac{7}{2}}-\frac{3}{16}\partial^2 O_{\frac{7}{2}}\right)(w)=\frac{1}{(z-w)^3}\ \left[-\frac{17}{8\sqrt{6}}P^{uu}_{4'}+\frac{17}{4}\sqrt{\frac{3}{2}}P^{ww}_{4'}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(GP^{uu}_{4'}-\frac{4\sqrt{6}}{9}\partial O_{\frac{9}{2}}-\frac{\sqrt{6}}{56}\partial^2 O_{\frac{7}{2}}\right)(w)=$$

$$\frac{1}{(z-w)^3}\ \left[\frac{(2574+2151k+239k^2)}{36(3+k)(6+k)}P^{uu}_{4'}+\frac{17}{6}P^{ww}_{4'}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(GP^{ww}_{4'}-\frac{2}{9}\sqrt{\frac{2}{3}}\partial O_{\frac{9}{2}}-\frac{1}{14}\sqrt{\frac{2}{3}}\partial^2 O_{\frac{7}{2}}\right)(w)=$$

$$\frac{1}{(z-w)^3}\ \left[-\frac{17}{27}P^{uu}_{4'}+\frac{2(666+441k+49k^2)}{9(3+k)(6+k)}P^{ww}_{4'}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\partial P^{ww}_{4'}\right)(w)=\frac{1}{(z-w)^3}\left[\frac{378+333k+37k^2}{6(3+k)(6+k)}\right]O_{\frac{7}{2}}(w)$$

$$+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(TP^{uu}_{4'}-\frac{1}{6}\partial^2 P^{uu}_{4'}\right)(w)=\frac{1}{(z-w)^4}\,5\sqrt{\frac{3}{2}}\,O_{\frac{7}{2}}(w)$$

$$+\frac{1}{(z-w)^3}\ \left[(-\frac{1}{7})5\sqrt{\frac{3}{2}}\,\partial O_{\frac{7}{2}}+13\sqrt{\frac{2}{3}}O_{\frac{9}{2}}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(TP^{ww}_{4'}-\frac{1}{6}\partial^2 P^{ww}_{4'}\right)(w)=\frac{1}{(z-w)^4}\,10\sqrt{\frac{3}{2}}\,O_{\frac{7}{2}}(w)$$

$$+\frac{1}{(z-w)^3}\left[(-\frac{1}{7})10\sqrt{\frac{3}{2}}\,\partial O_{\frac{7}{2}}+\frac{13}{3\sqrt{6}}O_{\frac{9}{2}}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\,\left(G\partial O_{\frac{7}{2}}-\frac{7}{3}\partial GO_{\frac{7}{2}}+\frac{1}{9\sqrt{6}}\partial^2 P^{uu}_{4'}-\frac{1}{3}\sqrt{\frac{2}{3}}\partial^2 P^{ww}_{4'}\right)(w)=$$

$$-\frac{1}{(z-w)^4}\left[\frac{7(54+99k+11k^2)}{3(3+k)(6+k)}\right]O_{\frac{7}{2}}(w)-\frac{1}{(z-w)^3}\,(-\frac{1}{7})\left[\frac{7(54+99k+11k^2)}{3(3+k)(6+k)}\right]\partial O_{\frac{7}{2}}(w)$$
$$+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\,\left(GO_{\frac{9}{2}}-\frac{2}{63}\sqrt{\frac{2}{3}}\partial^2 P^{uu}_{4'}+\frac{1}{21\sqrt{6}}\partial^2 P^{ww}_{4'}\right)(w)=\frac{1}{(z-w)^3}\left[\frac{(1422+927k+103k^2)}{9(3+k)(6+k)}\right]O_{\frac{9}{2}}(w)$$
$$+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\,\left(TO_{\frac{9}{2}}-\frac{3}{20}\partial^2 O_{\frac{9}{2}}\right)(w)=\frac{1}{(z-w)^4}\,\left[\frac{208}{35}\sqrt{\frac{2}{3}}P^{uu}_{4'}-\frac{52}{35}\sqrt{6}\,P^{ww}_{4'}\right](w)$$

$$+\frac{1}{(z-w)^3}\,(-\frac{1}{8})\partial\left[\frac{208}{35}\sqrt{\frac{2}{3}}P^{uu}_{4'}-\frac{52}{35}\sqrt{6}\,P^{ww}_{4'}\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\,\left(T\partial O_{\frac{7}{2}}-\frac{7}{4}\partial TO_{\frac{7}{2}}-\frac{1}{9}\partial^3 O_{\frac{7}{2}}\right)(w)=\frac{1}{(z-w)^4}\,\left[\frac{25}{6\sqrt{6}}\,P^{uu}_{4'}-\frac{25}{\sqrt{6}}\,P^{ww}_{4'}\right](w)$$

$$+\frac{1}{(z-w)^3}\,\left[(-\frac{1}{8})\frac{25}{6\sqrt{6}}\,\partial P^{uu}_{4'}-(-\frac{1}{8})\frac{25}{\sqrt{6}}\,\partial P^{ww}_{4'}\right.$$
$$\left.-\frac{21}{4}\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\,\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\,\partial P^{ww}_{4'}\right)\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\,\left(G\partial P^{uu}_{4'}-\frac{8}{3}\partial GP^{uu}_{4'}-\frac{\sqrt{6}}{5}\partial^2 O_{\frac{9}{2}}-\frac{2}{63}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right)(w)=$$

$$\frac{1}{(z-w)^4}\,\left[-\frac{4(2178+3249k+361k^2)}{45(3+k)(6+k)}\,P^{uu}_{4'}+\frac{56}{15}\,P^{ww}_{4'}\right](w)$$

$$+\frac{1}{(z-w)^3}\,\left[-(-\frac{1}{8})\,\frac{4(2178+3249k+361k^2)}{45(3+k)(6+k)}\partial P^{uu}_{4'}+(-\frac{1}{8})\frac{56}{15}\,\partial P^{ww}_{4'}\right.$$
$$\left.-2\sqrt{6}\,\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\,\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\,\partial P^{ww}_{4'}\right)\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\,\left(G\partial P^{ww}_{4'}-\frac{8}{3}\partial GP^{ww}_{4'}-\frac{1}{5\sqrt{6}}\partial^2 O_{\frac{9}{2}}-\frac{8}{189}\sqrt{\frac{2}{3}}\partial^3 O_{\frac{7}{2}}\right)(w)=$$

$$\frac{1}{(z-w)^4}\,\left[-\frac{112}{135}\,P^{uu}_{4'}-\frac{8(522+1341k+149k^2)}{45(3+k)(6+k)}\,P^{ww}_{4'}\right](w)$$

$$+\frac{1}{(z-w)^3}\,\left[-(-\frac{1}{8})\frac{112}{135}\,\partial P^{uu}_{4'}-(-\frac{1}{8})\frac{8(522+1341k+149k^2)}{45(3+k)(6+k)}\,\partial P^{ww}_{4'}\right.$$

$$-8\sqrt{\frac{2}{3}}\left(GO_{\frac{7}{2}}+\frac{1}{4\sqrt{6}}\,\partial P^{uu}_{4'}-\frac{\sqrt{6}}{4}\,\partial P^{ww}_{4'}\right)\Big](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(T\partial U-\frac{5}{4}\partial TU-\frac{1}{7}\partial^3 U\right)(w)=-\frac{1}{(z-w)^4}\,\frac{33}{7}\sqrt{\frac{3}{2}}\,W(w)$$

$$+\frac{1}{(z-w)^3}\left[-(-\frac{1}{6})\frac{33}{7}\sqrt{\frac{3}{2}}\partial W-\frac{15}{4}\left(GU-\frac{\sqrt{6}}{3}\partial W\right)\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(G\partial W-2\partial GW-\frac{1}{21}\sqrt{\frac{2}{3}}\partial^3 U\right)(w)=-\frac{1}{(z-w)^4}\left[\frac{4(18+261k+29k^2)}{7(3+k)(6+k)}\right]W(w)$$

$$+\frac{1}{(z-w)^3}\left[-(-\frac{1}{6})\left[\frac{4(18+261k+29k^2)}{7(3+k)(6+k)}\right]\partial W-5\sqrt{\frac{2}{3}}\left(GU-\frac{\sqrt{6}}{3}\partial W\right)\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(GO_{4''}-\frac{2}{9}\partial O_{\frac{9}{2}'}\right)(w)=\frac{1}{(z-w)^3}\left[\frac{16(3+2k)(15+2k)}{9(3+k)(6+k)}\right]O_{4''}(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(G\partial^2 U-4\partial G\partial U+\frac{5}{2}\partial^2 GU-\frac{1}{2\sqrt{6}}\partial^3 W\right)(w)=\frac{1}{(z-w)^5}\left[\frac{5(18+153k+17k^2)}{(3+k)(6+k)}\right]U(w)$$

$$+\frac{1}{(z-w)^4}(-\frac{2}{5})\left[\frac{5(18+153k+17k^2)}{(3+k)(6+k)}\right]\partial U(w)$$

$$+\frac{1}{(z-w)^3}\left[(\frac{1}{30})\left[\frac{5(18+153k+17k^2)}{(3+k)(6+k)}\right]\partial^2 U-2\sqrt{6}\left(GW-\frac{1}{6\sqrt{6}}\partial^2 U\right)\right.$$

$$\left.+10\left(TU-\frac{1}{4}\partial^2 U\right)\right](w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(T\partial W-\frac{3}{2}\partial TW-\frac{1}{8}\partial^3 W\right)(w)=-\frac{1}{(z-w)^5}\,\frac{5}{2}\sqrt{\frac{3}{2}}\,U(w)-\frac{1}{(z-w)^4}(-\frac{2}{5})\frac{5}{2}\sqrt{\frac{3}{2}}\partial U(w)$$

$$+\frac{1}{(z-w)^3}\left[-(\frac{1}{30})\frac{5}{2}\sqrt{\frac{3}{2}}\partial^2 U-\frac{9}{2}\left(GW-\frac{1}{6\sqrt{6}}\partial^2 U\right)+5\sqrt{\frac{2}{3}}\left(TU-\frac{1}{4}\partial^2 U\right)\right](w)$$

$$+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(TO_{4''}-\frac{1}{6}\partial^2 O_{4''}\right)(w)=\frac{1}{(z-w)^3}\,\frac{13}{6}O_{\frac{9}{2}'}(w)+\mathcal{O}((z-w)^{-2}),$$

$$G(z)\ \left(GO_{\frac{9}{2}'}-\frac{1}{9}\partial^2 O_{4''}\right)(w)=\frac{1}{(z-w)^3}\,\frac{(1422+927k+103k^2)}{9(3+k)(6+k)}O_{\frac{9}{2}'}(w)+\mathcal{O}((z-w)^{-2}).\quad \text{(E.1)}$$

One sees, in (E.1), the k-dependence for the quasi-primary fields which contain $G(w)$ while one does not see the k-dependence for the remaining quasi-primary fields. This is obvious because the OPE between $G(z)$ and $G(w)$ has the explicit c-dependence from (2.2). One can check that the coefficients (C.2) hold in this case.

References

[1] M. R. Gaberdiel and R. Gopakumar, "An AdS_3 Dual for Minimal Model CFTs," Phys. Rev. D **83**, 066007 (2011) [arXiv:1011.2986 [hep-th]].

[2] M. R. Gaberdiel and R. Gopakumar, "Triality in Minimal Model Holography," JHEP **1207**, 127 (2012) [arXiv:1205.2472 [hep-th]].

[3] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, "Coset Construction for Extended Virasoro Algebras," Nucl. Phys. B **304**, 371 (1988).

[4] C. Ahn, "The Coset Spin-4 Casimir Operator and Its Three-Point Functions with Scalars," JHEP **1202**, 027 (2012) [arXiv:1111.0091 [hep-th]].

[5] M. R. Gaberdiel and R. Gopakumar, "Minimal Model Holography," arXiv:1207.6697 [hep-th].

[6] R. Gopakumar, A. Hashimoto, I. R. Klebanov, S. Sachdev and K. Schoutens, "Strange Metals in One Spatial Dimension," Phys. Rev. D **86**, 066003 (2012) [arXiv:1206.4719 [hep-th]].

[7] M. R. Douglas, "G / H Conformal Field Theory," CALT-68-1453.

[8] K. Schoutens and A. Sevrin, "Minimal superW(N) algebras in coset conformal field theories," Phys. Lett. B **258**, 134 (1991).

[9] C. Ahn, K. Schoutens and A. Sevrin, "The full structure of the super W(3) algebra," Int. J. Mod. Phys. A **6**, 3467 (1991).

[10] K. Hornfeck and E. Ragoucy, "A Coset Construction For The Super W(3) Algebra," Nucl. Phys. B **340**, 225 (1990).

[11] T. Inami, Y. Matsuo and I. Yamanaka, "Extended Conformal Algebras With N=1 Supersymmetry," Phys. Lett. B **215**, 701 (1988).

[12] C. Ahn, "The Large N 't Hooft Limit of Coset Minimal Models," JHEP **1110**, 125 (2011) [arXiv:1106.0351 [hep-th]].

[13] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, "Extensions of the Virasoro Algebra Constructed from Kac-Moody Algebras Using Higher Order Casimir Invariants," Nucl. Phys. B **304**, 348 (1988).

[14] P. Bowcock, "Quasi-primary Fields And Associativity Of Chiral Algebras," Nucl. Phys. B **356**, 367 (1991).

[15] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel and R. Varnhagen, "W algebras with two and three generators," Nucl. Phys. B **361**, 255 (1991).

[16] P. Bouwknegt and K. Schoutens, "W symmetry in conformal field theory," Phys. Rept. **223**, 183 (1993) [hep-th/9210010].

[17] W. Nahm, "Algebras of two-dimensional chiral fields and their classification," In *Islamabad 1989, Proceedings, Mathematical physics* 283-300. (see Conference Index)

[18] W. Nahm, "Chiral algebras of two-dimensional chiral field theories and their normal ordered products," In *Trieste 1989, Proceedings, Recent developments in conformal field theories* 81-84. (see Conference Index)

[19] T. Creutzig, Y. Hikida and P. B. Ronne, "Higher spin AdS_3 supergravity and its dual CFT," JHEP **1202**, 109 (2012) [arXiv:1111.2139 [hep-th]].

[20] C. Candu and M. R. Gaberdiel, "Supersymmetric holography on AdS_3," arXiv:1203.1939 [hep-th].

[21] M. Henneaux, G. Lucena Gomez, J. Park and S. -J. Rey, "Super- W(infinity) Asymptotic Symmetry of Higher-Spin AdS_3 Supergravity," JHEP **1206**, 037 (2012) [arXiv:1203.5152 [hep-th]].

[22] K. Hanaki and C. Peng, "Symmetries of Holographic Super-Minimal Models," arXiv:1203.5768 [hep-th].

[23] C. Ahn, "The Large N 't Hooft Limit of Kazama-Suzuki Model," JHEP **1208**, 047 (2012) [arXiv:1206.0054 [hep-th]].

[24] C. Candu and M. R. Gaberdiel, "Duality in N=2 minimal model holography," arXiv:1207.6646 [hep-th].

[25] C. Ahn, "The Operator Product Expansion of the Lowest Higher Spin Current at Finite N," arXiv:1208.0058 [hep-th].

[26] T. Creutzig, Y. Hikida and P. B. Ronne, "N=1 supersymmetric higher spin holography on AdS_3," arXiv:1209.5404 [hep-th].

[27] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, "Black holes in three dimensional higher spin gravity: A review," arXiv:1208.5182 [hep-th].

[28] E. Perlmutter, T. Prochazka and J. Raeymaekers, "The semiclassical limit of W_N CFTs and Vasiliev theory," arXiv:1210.8452 [hep-th].

[29] J. R. David, M. Ferlaino and S. P. Kumar, "Thermodynamics of higher spin black holes in 3D," arXiv:1210.0284 [hep-th].

[30] T. Creutzig and A. R. Linshaw, "The super $W_{1+\infty}$ algebra with integral central charge," arXiv:1209.6032 [math.QA].

[31] B. Chen, J. Long and Y. -n. Wang, "Black holes in Truncated Higher Spin AdS_3 Gravity," arXiv:1209.6185 [hep-th].

[32] M. Taronna, "Higher-Spin Interactions: three-point functions and beyond," arXiv:1209.5755 [hep-th].

[33] S. Banerjee, A. Castro, S. Hellerman, E. Hijano, A. Lepage-Jutier, A. Maloney and S. Shenker, "Smoothed Transitions in Higher Spin AdS Gravity," arXiv:1209.5396 [hep-th].

[34] P. Kraus and E. Perlmutter, "Probing higher spin black holes," arXiv:1209.4937 [hep-th].

[35] S. Lal and B. Sahoo, "Holographic Renormalisation for the Spin-3 Theory and the (A)dS3/CFT2 correspondence," arXiv:1209.4804 [hep-th].

[36] H. Afshar, M. Gary, D. Grumiller, R. Rashkov and M. Riegler, "Non-AdS holography in 3-dimensional higher spin gravity - General recipe and example," arXiv:1209.2860 [hep-th].

[37] I. Fujisawa and R. Nakayama, "Second-Order Formalism for 3D Spin-3 Gravity," arXiv:1209.0894 [hep-th].

[38] S. Fredenhagen and C. Restuccia, "The geometry of the limit of N=2 minimal models," arXiv:1208.6136 [hep-th].

[39] H. S. Tan, "Exploring Three-dimensional Higher-Spin Supergravity based on sl(N —N - 1) Chern-Simons theories," arXiv:1208.2277 [hep-th].

[40] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, "Towards metric-like higher-spin gauge theories in three dimensions," arXiv:1208.1851 [hep-th].

[41] K. Thielemans, “A Mathematica package for computing operator product expansions,” Int. J. Mod. Phys. C **2**, 787 (1991).

[42] D. Friedan, E. J. Martinec and S. H. Shenker, “Conformal Invariance, Supersymmetry and String Theory,” Nucl. Phys. B **271**, 93 (1986).

[43] C. Ahn, “The Primary Spin-4 Casimir Operators in the Holographic SO(N) Coset Minimal Models,” JHEP **1205**, 040 (2012) [arXiv:1202.0074 [hep-th]].

[44] K. Thielemans, “An Algorithmic approach to operator product expansions, W algebras and W strings,” hep-th/9506159.

[45] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck and R. Hubel, “Coset realization of unifying W algebras,” Int. J. Mod. Phys. A **10**, 2367 (1995) [hep-th/9406203].